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Relative positioning by navigation satellite using Doppler shifts and propagation delays as observables

Lorys Ascher
Iowa State University

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I. SATELLITE NAVIGATION SYSTEMS AND RELATIVE POSITIONING

A. Satellite Navigation

The Navy Navigation Satellite System (NNSS), also called Transit, has been operational since 1964 and uses four satellites in circular polar orbits at 600 miles of altitude. Each satellite transmits a stable 400 MHz continuous wave which is phase modulated with binary data describing the satellite trajectory and timing markers.

Anywhere on earth one can track a satellite pass and get a position fix every 1 1/2 to 2 hours. The relative velocity of the satellite with respect to the earth surface creates a Doppler shift in the 400 MHz signal when it is received by a ground station. The received signal is compared with that of a ground oscillator, the difference between them being integrated for each period of about 20 seconds between timing marks, thus generating a sequence of Doppler counts. This sequence is determined by the satellite path, the location of the receiver on earth, and both oscillator offsets. A least square fitting technique is used to find which ground position and oscillator offsets best explain the observed sequence of counts and gives their estimates (Fig. 1.1). A more complete description of the Transit system can be found in (13).

Other satellite navigation schemes have been studied (14). An attractive one involves stationary satellites which could be used for other purposes as well, such as relays. Stationary satellites being higher (about 20,000 miles altitude) also provide a good basis for triangulations for the navigation of space ships in the vicinity of the earth in addition to sea level or low altitude aircraft positioning.

Stationary satellites do not generate Doppler shifts in the signals received on the ground, and therefore the range measurements have to be obtained directly from time measurements. We show here that this method of ranging by time measurements could be used to improve the present Transit receivers and is very advantageous for relative positioning.

At present, one advantage of Transit is that it exists and is maintained by the United States Navy and free for the other users. Another advantage is that it is a passive system since it does not require the satellite to respond to a particular user therefore not limiting the number of users and completely separating the Navy's responsibilities from the user's.

B. Relative Positioning (Translocation, Fig. 1.2)

If two receivers are close enough to track the same satellite pass, high accuracy in the relative position estimate can be expected because of crosscorrelation in the absolute position errors and also the fact that satellite position errors nearly cancel out (16). In surveying applications the data (Doppler counts) do not have to be processed immediately and could be processed later on a large computer, thus simplifying the receivers by suppressing the small computers normally found on Transit receivers. So far (16) translocation has been done with two conventional receivers using Doppler counts only for ranging. It still would be possible to use accurate atomic clocks in both receivers to measure the time of arrival of the markers sent by the satellite and use the time lag between them in addition to Doppler counts to find the relative positions of the receivers. The time lag between arrival of the same marker in both receivers is directly proportional to their relative distance and should be advantageous for relative positioning. Such a system does not require any change in the present Transit system as far as the satellite and all the Navy's tracking stations are concerned. Only the receivers need to be changed.

In this research we evaluate the improvement obtained in both absolute and relative positioning using time measurements in addition to Doppler counts.

The performance of a system using time measurements only is evaluated as well as the performance of a simplified suboptimal version of it.

C. Surveying

Surveying using electromagnetic waves (without a satellite) involves receiving signals from two ground transmitters also being used as a triangulation basis. They are divided in circular, hyperbolic, or elliptical systems depending if the sensors respond to distance, distance difference, or distance sum respectively. These systems, for ranges of about 100 miles, have a relative accuracy of about 30 feet (7). More accurate distance measurements can be made using higher carrier frequencies but require a direct line of sight between transmitter and receiver, which is often not practical.

Using a satellite pass is then equivalent to having it act as a succession of transmitters which are used for triangulation provided the satellite path is known with sufficient accuracy. Then the translocation system can be used for surveying purposes and give the relative position of one receiver with respect to the other in terms of altitude, latitude, and longitude without the need of a direct line of sight and without need of a third piece of equipment to triangulate with.

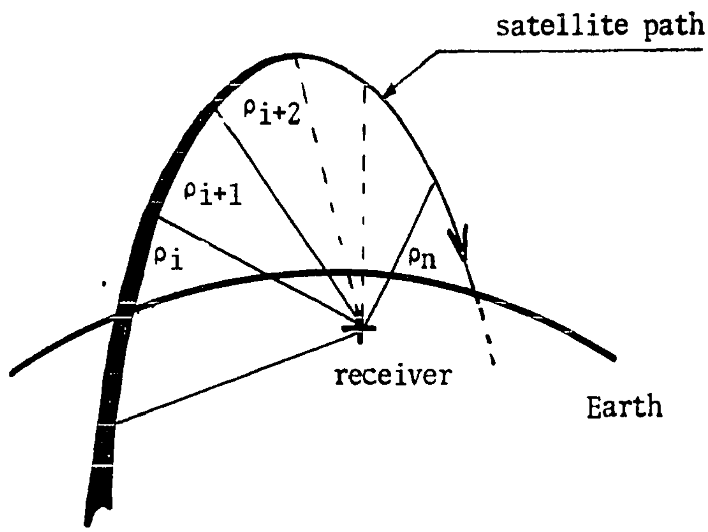

Fig. 1.1. Satellite navigation

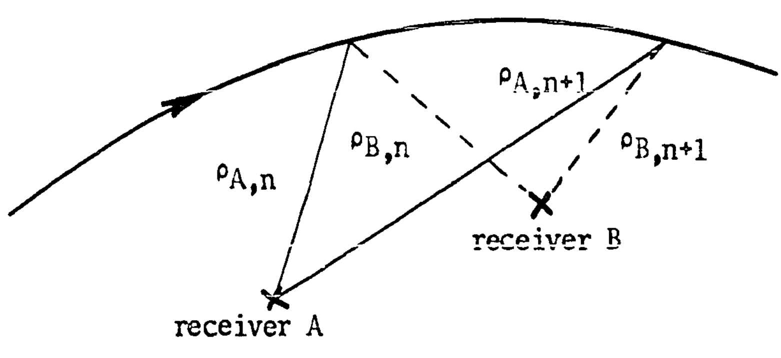

Fig. 1.2. Relative positioning

II. SYSTEM INTEGRATION

A. Introduction

In this chapter some results on Kalman filtering are recapitulated, and the coordinate systems used are defined and used to get the dynamic model and the measurement model. Miscellaneous parameters necessary for the Kalman filter are also defined. Last, the modeling of a simplified translocation system is explained.

The observables used to estimate absolute and relative positions of the ground receivers are the measured Doppler counts and the times of arrival of the markers in addition to the predicted satellite trajectory (Fig. 2.1). The observables are nonlinear functions of the receiver's positions, satellite position, and also oscillator frequencies and clock synchronization error. In order to have a linear model the estimation procedure is not done on the actual measurements. Rather the position errors are estimated by comparing the actual measurements with computed "measurements" based on the original reference estimates of receivers and satellite positions, and oscillators and clock offsets.

Two modes of operation can exist.

In the open loop mode computed measurements are always based on the same original reference estimates. The final estimates being equal to the original reference estimates minus the estimated errors.

In the closed loop mode the reference estimates on which the computed measurements are based are updated at each step making the reference estimates closer to the true values and thus reducing the interval of linearization of each estimated variable. There is no analytical way

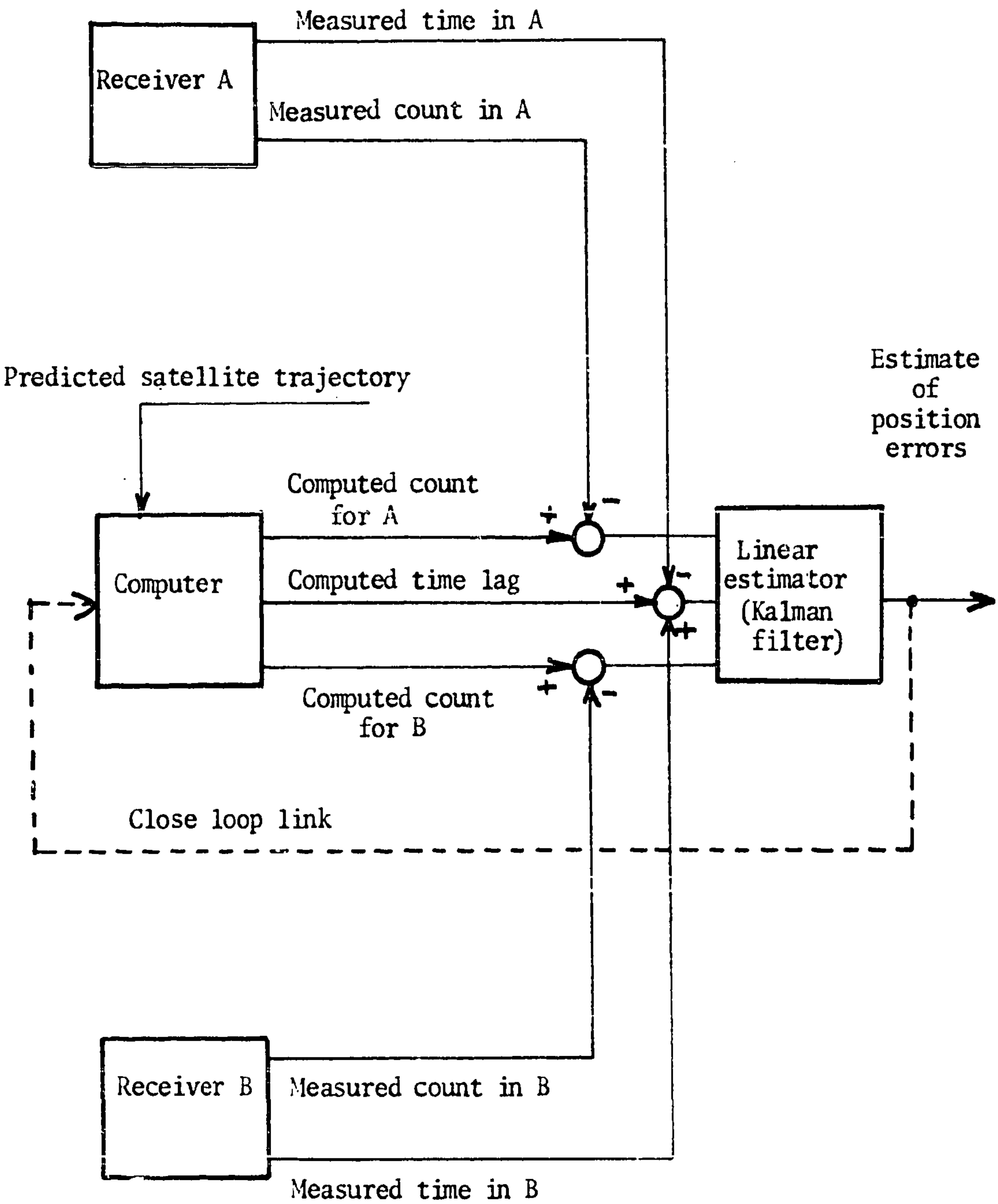

Fig. 2.1. Block diagram for translocation

of knowing if the closed loop mode of operation is stable and converges. If stable, it should be, on the average, more accurate than the open loop system since the approximation due to the linearization is reduced at each step, while in the open loop mode the linearization remains only as good as the original estimates. This last statement is only a heuristic argument which makes sense from an engineering view point, and there is no known analytical way to prove it. Only a Monte Carlo simulation could give an idea of the closed loop operation in terms of stability and accuracy. Simulating the open loop mode should give an upper bound for the average estimation error of the closed loop system should it be stable. In order to save computer time a variance analysis of the open loop mode is made rather than a Monte Carlo simulation. For the open loop mode the only advantage of a Monte Carlo simulation would be to give an idea of the errors caused by the linearization but this is already known to be negligible from existing navigation systems which use the same type of linearized equations.

Because of its convenience for computer implementation in a real life system, a Kalman filter is used for the estimator.

B. Kalman Filter

Kalman filter theory is adequately treated elsewhere so only the salient aspects will be mentioned here.

1. Standard Kalman filter

The process to be estimated is assumed to satisfy the vector differential equation

$$\dot{x} = A(t)x + u(t) \tag{2.1}$$

where x = System state vector

$A(t)$ = Dynamics matrix

$u(t)$ = White noise input vector

Nonwhite processes are modeled by having a shaping filter act on a white noise, as shown by Sorenson (12) and Brown (1), thus augmenting the size of the matrix A and fitting the above model.

Discretizing (2.1) we get

$$x_{n+1} = \phi_n x_n + g_n \tag{2.2}$$

where x_n = State vector at time t_n

ϕ_n = Transition matrix

g_n = Response to white noise input vector in interval t_n to t_{n+1}

The inputs (data) for the Kalman filter are discrete measurements of the form

$$y_n = M_n x_n + \delta y_n \tag{2.3}$$

where y_n = Measurement vector at time t_n

M_n = Measurement matrix at time t_n

δy_n = Time uncorrelated measurement error vector

Assuming all the above, Kalman (9) has shown that $\hat{x}_n$, the minimum mean square error estimate of x_n, is given by

$$\hat{x}_n = \hat{x}'_n + b_n(y_n - M_n\hat{x}'_n) \tag{2.4}$$

with error covariance matrix $P_n = E(\hat{x}_n - x_n)(\hat{x}_n - x_n)^T$

given by $$P_n = P_n^* - b_n(M_n P_n^* M_n^T + V_n)b_n^T \tag{2.5}$$

where $$b_n = P_n^* M_n^T (M_n P_n^* M_n^T + V_n)^{-1} \tag{2.6}$$

$$\hat{x}'_n = \phi_{n-1}\hat{x}_{n-1} \tag{2.7}$$

$$P_n^* = \phi_{n-1}P_{n-1}\phi_{n-1}^T + H_{n-1} \tag{2.8}$$

$$H_{n-1} = E(g_{n-1}g_{n-1}^T) \tag{2.9}$$

$$V_n = E(\delta y_n \delta y_n^T) \tag{2.10}$$

The above equations give a recursive way to get the best estimate of x_n and the corresponding estimation error on the basis of the last estimate of the state vector $\hat{x}_{n-1}$ and its error covariance matrix P_{n-1} the new measurement vector y_n, and its known connection with the state vector (i.e. the matrix M_n), and measurement error statistic V_n. All other needed parameters are intrinsic to the dynamic model (2.1) and (2.2).

In practice, the dynamic model is known before hand even though the knowledge of ϕ_{n-1} and H_{n-1} is only needed at time t_n in order to get $\hat{x}_n$.

The same applies to the measurement model (M_n and V_n) which is very useful in practice since any new measurement of any linear combination of the state components can be used. This permits the use of new sources of "information" as they occur without needing prior knowledge of their occurrences and relationships to the states. The limitations to this versatility are due to programming limitations, not to the Kalman algorithm itself.

The above one step equations require $t_n \geq t_{n-1}$. The case $t_n = t_{n-1}$ ($\phi = I$) corresponds to re-updating the estimate using a new measurement synchronous with the last one used and such that their errors are not cross correlated. This permits simplification of the computations in the case of a high dimension measurement vector if it can be broken down into several measurements with independent errors and processing each measurement sequentially (12).

The above equations do not work for $t_n < t_{n-1}$ (smoothing) but a recursive Kalman algorithm does exist (1).

If the recursive procedure is started with $\hat{x}_0 = E(x_0)$ and $P_0 = E(x_0 x_0^T)$ then the estimate $\hat{x}_n$ is unbiased.

Kalman filter using a gain b_n different from the optimal given by (2.6) is suboptimal, and the associated error covariance is then obtained by replacing (2.5) by:

$$P_n = (I - b_n M_n) P_n^* (I - b_n M_n)^T + b_n V_n b_n^T \tag{2.11}$$

2. <u>Delayed state Kalman filter</u>

In some applications, processing of Doppler counts for instance, the measurement vector is a linear combination of both present and previous state vector. Or:

$$y_n = M_n x_n + N_n x_{n-1} + \delta y_n \tag{2.12}$$

A Kalman filter for this model is given by Brown and Hartman in (3). Stuva (15) derived an equivalent algorithm that is less sensitive to round off errors in the case of Doppler counts applications.

Equations (2.2) and (2.12) describe the model.

The recursive equations for Stuva's algorithm are:

$$b_n = [\phi_{n-1} P_{n-1} (M_n \phi_{n-1} + N_n)^T + H_{n-1} M_n^T] Q_n^{-1} \tag{2.13}$$

$$\hat{x}_n = \phi_{n-1} \hat{x}_{n-1} + b_n [y_n - (M_n \phi_{n-1} + N_n) \hat{x}_{n-1}] \tag{2.14}$$

$$P_n = \phi_{n-1} P_{n-1} \phi_{n-1}^T + H_{n-1} - b_n Q_n b_n^T \tag{2.15}$$

where

$$Q_n = (M_n \phi_{n-1} + N_n) P_{n-1} (M_n \phi_{n-1} + N_n)^T + V_n + M_n H_{n-1} M_n^T \tag{2.16}$$

C. Coordinate Systems

The coordinate systems used are shown on Fig. 2.2.

The coordinate system used throughout to define positions of receivers and satellite and used to define the state variables is earth-fixed polar. An absolute (inertial) coordinate system is not necessary here since this study does not involve sensors responding to accelerations.

Two other coordinate systems are used only for the computations related to geometry in the measurement model. They are the earth-fixed rectangular coordinate system and the local rectangular coordinate system which permits the definition of direction cosines.

The local rectangular system is also used in the last step of each simulation to convert position uncertainties in latitude and longitude, expressed in radians, to position uncertainties in feet in the east-west and north-south directions.

D. Dynamic Model

1. Introduction

For implementation of the Kalman filter the dynamic model includes states for the ground receivers' positions, satellite coordinates and states for oscillators and clocks errors.

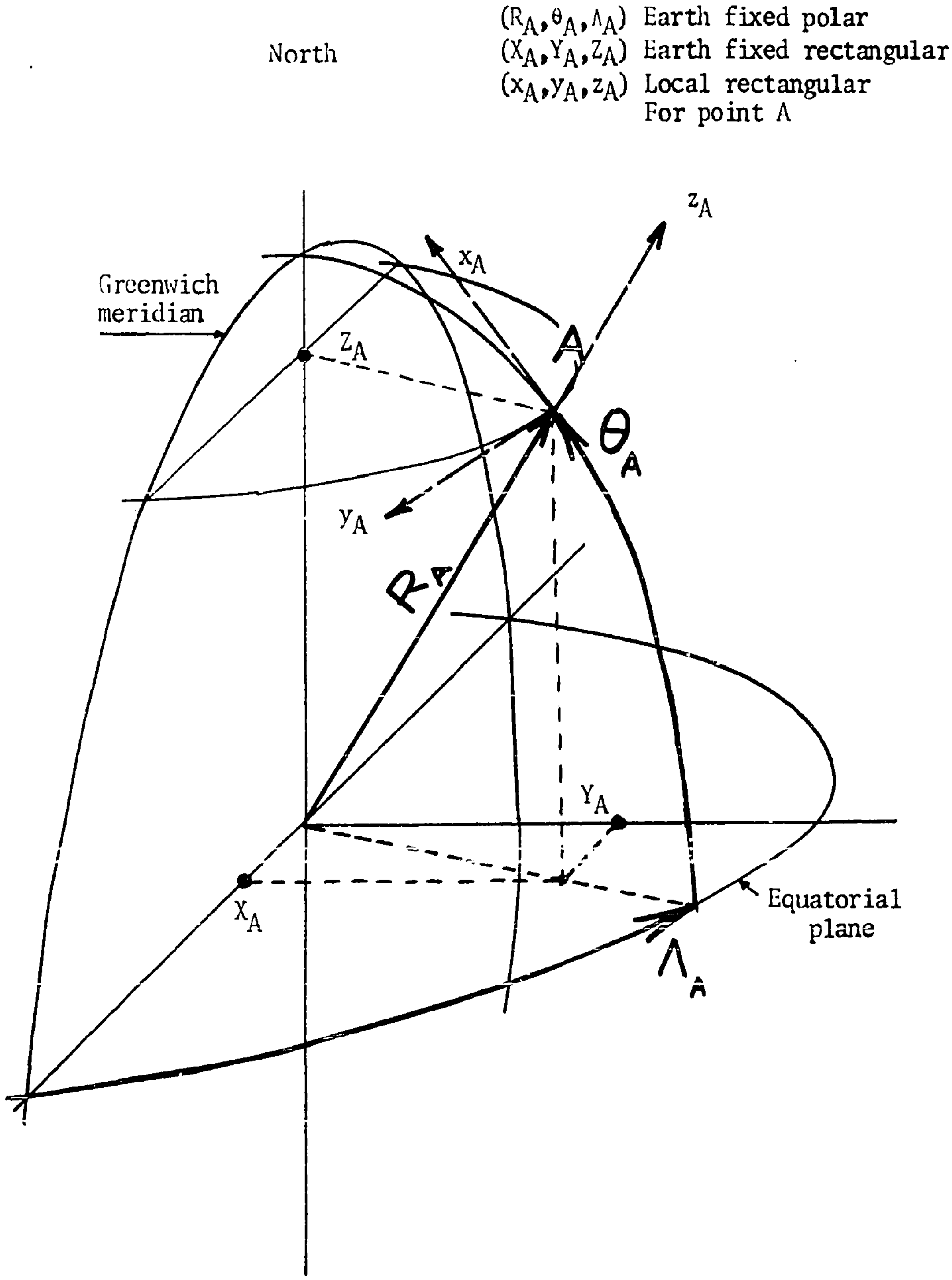

Fig. 2.2. Coordinate systems

The satellite position errors (deviations from the predicted path) are assumed to be harmonic of period equal to the time of revolution of the satellite. This would be unrealistic if the same satellite was to be used for several successive passes as the satellite position errors are mainly caused by a lack of knowledge of the earth gravity field. Since in the simulation we use the same satellite only once and since the time the satellite is tracked is short compared to one period this simple way of simulating the position errors of the satellite does not affect the validity of the results.

The oscillator errors and clocks errors are modeled using the shaping filter technique.

2. Receivers position errors

δR_A, $\delta\theta_A$, $\delta\Lambda_A$ are coordinates errors for receiver A.

δR_B, $\delta\theta_B$, $\delta\Lambda_B$ are coordinates errors for receiver B.

These states are modeled as random constant biases. Thus

$$\dot{x}_i = 0 \tag{2.17}$$

3. Satellites position errors

δR_S, $\delta\theta_S$, $\delta\Lambda_S$ are coordinates errors of the satellite.

These states are modeled as harmonic processes of independent random amplitude and phases of period equal to the time of one satellite revolution. Each satisfies the differential equation:

$$\ddot{x} + \omega^2 x = 0 \tag{2.18}$$

or in state form:

$$\begin{bmatrix} \dot{x}_i \\ \dot{x}_{i+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} x_i \\ x_{i+1} \end{bmatrix} \tag{2.19}$$

Since we assume phase and amplitude to be independent we have for initial condition:

$$E[x(o)\dot{x}(0)] = 0 \tag{2.20}$$

4. Receivers and satellites oscillators time correlated errors

The frequency errors of the satellite oscillators are modeled as independent first-order Gaussian Markov processes. Each is generated by a shaping filter (4 and 12), acting on a white noise driving function, whose input-output differential equation in state form is:

$$\dot{x}_i = -\beta_i x_i + \sqrt{2\sigma_i^2 \beta_i} \quad f_i \tag{2.21}$$

where f_i = Unit white noise

β_i = Inverse time constant of Markov process

$\sigma_i^2 = E[x_i^2]$ = Variance of frequency error

5. Time measurement correlated error

The error on the time measurement of arrival of markers in both receivers A and B is modeled as an integrated white noise (random walk). In state form

$$\dot{x}_i = \sigma_i f_i \tag{2.22}$$

6. Dynamic model

We can now get the plant equation by defining the states:

$x_1 = \delta R_A$	feet	ground receiver A
$x_2 = \delta\theta_A$	radians	
$x_3 = \delta\Lambda_A$	radians	
$x_4 = \delta R_B$	feet	ground receiver B
$x_5 = \delta\theta_B$	radians	
$x_6 = \delta\Lambda_B$	radians	

$x_7 = \delta R_S$	feet	satellite	
$x_8 = \dot{x}_7$	feet/second		
$x_9 = \delta\theta_S$	radians		
$x_{10} = \dot{x}_9$	radians/second		
$x_{11} = \delta\Lambda_S$	radians		
$x_{12} = \dot{x}_{11}$	radians/second		
$x_{13} = \delta f_A$	Hertz	oscillator in receiver A	
$x_{14} = \delta f_B$	Hertz	oscillator in receiver B	
$x_{15} = \delta f_S$	Hertz	satellite oscillator	
$x_{16} = \delta\Lambda_M$	seconds	clocks' synchronization error	

Now, let the entire state model be

$$\dot{x} = Ax + u$$

The nonzero elements of the matrix A are then

$$a_{7,8} = a_{9,10} = a_{11,12} = 1$$

$$a_{8,7} = a_{10,9} = a_{12,11} = -\omega^2$$

$$a_{13,13} = -\beta_A$$

$$a_{14,14} = -\beta_B$$

$$a_{15,15} = -\beta_S$$

where $\omega = \frac{2\pi}{T}$ and T is the period of the satellite.

The nonzero driving terms are

$$u_{13} = \sqrt{2\sigma_A^2\beta_A}\ f_{13}$$

$$u_{14} = \sqrt{2\sigma_B^2\beta_B}\ f_{14}$$

$$u_{15} = \sqrt{2\sigma_S^2\beta_S}\ f_{15}$$

$$u_{16} = \sigma_C\ f_{16}$$

where f_i's are independent unit white noises.

This completes the dynamic model.

E. Measurement Model

1. Introduction

In order to use Kalman filtering the observables must be expressed as linear combinations of the state vector components or state vectors if the delayed state filter is to be used.

This linearization is done by first expressing the observables (Doppler counts and time lag) in terms of range or range rate between receiver and satellite. Then the relationship between a small variation of the observable and a corresponding variation of the range (ρ) is found. Also the linear relationship between a range variation ($\delta\rho$) and a coordinate variation ($\delta R, \delta\theta, \delta\Lambda$) at either end is found by differentiation. Finally, substituting, the variation of the observable is directly related through the linear relation to variations of coordinates at both end points of the range between satellite and ground receiver. These coordinates having been chosen as state variables, this last relation is the needed link for the measurement equation of the Kalman filter.

2. Linearization coefficients

The linearized equation relating $\delta\rho$ and δR, $\delta\Lambda$, $\delta\theta$ is obtained by partial differentiation of ρ with respect to R, θ and Λ and is given in Hartman and Brown (3).

$$\delta\rho = \frac{1}{\rho} \left[(R - R_S C_{zz_S}) \delta R + R R_S C_{yz_S} \delta\Lambda - R R_S C_{xz_S} \delta\theta \right] \qquad (2.23)$$

where R, θ, Λ are the ground receiver coordinates

R_S, θ_S, Λ_S are the satellite coordinates

C_{zz_S}, C_{yz_S}, C_{xz_S} are the direction cosines of ground local rectangular coordinate system x, y, z with respect to satellite local rectangular coordinate system x_S, y_S, z_S.

The same relation may be applied to find the variation of the range due to perturbations of the satellite coordinates R_S, θ_S, Λ_S:

$$\delta\rho = \frac{1}{\rho}\left[(R_S - RC_{z_S z})\delta R_S + R_S R C_{y_S z}\delta\Lambda_S - R_S R C_{x_S z}\delta\theta_S\right] \tag{2.24}$$

where $C_{z_S z}$, $C_{y_S z}$, $C_{x_S z}$ are the direction cosines of the satellite with respect to ground station.

Summing both linearized equations we get the total variation of the range due to perturbations of both satellite and ground station coordinates:

$$\delta\rho = A\delta R + B\delta\theta + C\delta\Lambda + D\delta R_S + E\delta\theta_S + F\delta\Lambda_S$$

where

$$A = \frac{R - R_S C_{zz_S}}{\rho} \qquad D = \frac{R_S - RC_{z_S z}}{\rho}$$

$$B = \frac{-RR_S C_{xz_S}}{\rho} \qquad E = \frac{-R_S R C_{x_S z}}{\rho}$$

$$C = \frac{RR_S C_{yz_S}}{\rho} \qquad F = \frac{R_S R C_{y_S z}}{\rho}$$

3. Equation for Doppler measurements

The measured Doppler count is proportional to the range difference and is given in Stansell (13).

$$N_m = (f - f_S)\Delta T + \frac{f}{c}\left[\rho(t_n) - \rho(t_{n-1})\right] + \delta N \tag{2.25}$$

where ΔT = counting interval

f = ground oscillator frequency

f_s = satellite oscillator frequency

δN = uncorrelated count error

$\rho(t)$ = actual range at time t

c = light velocity

The Doppler count predicted on the basis of the erroneous range $\rho + \delta\rho$ is:

$$N_c = [f+\delta f-f_s-\delta f_s]\Delta T+\frac{f}{c}[\rho(t_n)+\delta\rho(t_n)-\rho(t_{n-1})-\delta\rho(t_{n-1})] \quad (2.26)$$

where δf = ground oscillator error

δf_s = satellite oscillator error

$\delta\rho$ = range error

The input to the Kalman filter is:

$$N_c - N_m = \frac{f}{c}[\delta\rho(t_n) - \delta\rho(t_{n-1})] - \delta N - \Delta T\delta f + \Delta T\delta f_s \quad (2.27)$$

Replacing the $\delta\rho$'s by their linearizations in terms of the coordinate errors on ground station and satellite we get:

$$\begin{aligned} N_c - N_m = \frac{1}{\lambda} [&A_n\delta R_n + B_n\delta\theta_n + C_n\delta\Lambda_n \\ &+ D_n\delta R_{sn} + E_n\delta\theta_{sn} + F_n\delta\Lambda_{sn} \\ &- A_{n-1}\delta R_{n-1} - B_{n-1}\delta\theta_{n-1} - C_{n-1}\delta\Lambda_{n-1} \\ &- D_{n-1}\delta R_{s,n-1} - E_{n-1}\delta\theta_{s,n-1} - F_{n-1}\delta\Lambda_{s,n-1}] \\ &- \Delta T\delta f + \Delta T\delta f_s - \delta N \end{aligned} \quad (2.28)$$

where $A_n = A$ at $t = t_n$, $B_n = B$ at $t = t_n$, . . . etc.

and $A_{n-1} = A$ at $t = t_{n-1}$, $B_{n-1} = B$ at $t = t_{n-1}$, . . . etc.

These are the linearization coefficients corresponding to the geometries at times t_n and t_{n-1}.

and $\delta R_n = \delta R$ at $t = t_n, \delta\theta_n = \delta\theta$ at $t = t_n$, . . . etc.

$\delta R_{sn} = \delta R_s$ at $t = t_n$, . . . etc.

We get two such equations, one for each receiver with corresponding geometry coefficients.

4. Equation for time measurements

The time interval at which the same time mark transmitted from the satellite is received in receivers A and B is theoretically:

$$\tau = \frac{1}{c} [\rho_A - \rho_B] \tag{2.29}$$

ρ_A = actual range from satellite to receiver A

ρ_B = actual range from satellite to receiver B

c = light velocity

The measured time interval is:

$$\tau_m = \frac{1}{c} [\rho_A(t_n) - \rho_B(t_n)] + \Delta\tau + \delta\tau \tag{2.30}$$

$\Delta\tau$ = correlated time measurement error

$\delta\tau$ = uncorrelated time measurement error

The correlated error is here mainly the error in clock synchronization.

The predicted time interval is:

$$\tau_c = \frac{1}{c} [\rho_A(t_n) + \delta\rho_A(t_n) - \rho_B(t_n) - \delta\rho_B(t_n)] \tag{2.31}$$

The input to the Kalman filter is then:

$$\tau_c - \tau_m = \frac{1}{c} [\delta\rho_A(t_n) - \delta\rho_B(t_n)] - \Delta\tau - \delta\tau \tag{2.32}$$

Replacing the $\delta\rho$'s by their linearizations in terms of the states we get:

$$\tau_c - \tau_m = \frac{1}{c}\,[A_{An}\delta R_{An} + B_{An}\delta\theta_{An} + C_{An}\delta\Lambda_{An}$$
$$+ D_{An}\delta R_{Sn} + E_{An}\delta\theta_{Sn} + F_{An}\delta\Lambda_{Sn}$$
$$- A_{Bn}\delta R_{Bn} - B_{Bn}\delta\theta_{Bn} - C_{Bn}\delta\Lambda_{Bn}$$
$$- D_{Bn}\delta R_{Sn} - E_{Bn}\delta\theta_{Sn} - F_{Bn}\delta\Lambda_{Sn}]$$
$$- \Delta\tau - \delta\tau \tag{2.33}$$

Where A_{An}, B_{An}, C_{An} . . . . F_{An} are linearization coefficients for the range from receiver A to the satellite at time t_n. A_{Bn}, B_{Bn}, C_{Bn} . . . . . are linearization coefficients for the range from receiver B to satellite.

5. Measurement model

Knowing the measurement equations for Doppler counts and times we can define the measurement vector:

$$y = \begin{bmatrix} N_{cA}(t_n) - N_{mA}(t_n) \\ N_{cB}(t_n) - N_{mB}(t_n) \\ \tau_c(t_n) - \tau_m(t_n) \end{bmatrix} \tag{2.34}$$

where $N_{cA}(t_n) = N_c$ for receiver A at t_n

$N_{cB}(t_n) = N_c$ for receiver B at t_n

$N_{mA}(t_n) = N_m$ for receiver A at t_n

$N_{mB}(t_n) = N_m$ for receiver B at t_n

and letting $\frac{A}{\lambda} = a$, $\frac{B}{\lambda} = b$, ... $\frac{F}{\lambda} = f$

$\frac{A}{c} = \alpha$, $\frac{B}{c} = \beta$, ... $\frac{F}{c} = \varphi$

we get a delayed state measurement equation of the form:

$$y = M_n x_n + N_n x_{n-1} + v \tag{2.35}$$

where

$$v = \begin{bmatrix} \delta N_A \\ \delta N_B \\ \delta\tau \end{bmatrix} \qquad (2.36)$$

δN_A = uncorrelated count error in A

δN_B = uncorrelated count error in B

$\delta\tau$ = uncorrelated time error

The state equation and the measurement equation just derived are suitable for using a delayed state Kalman filter. The matrices M_n and N_n and N_n are shown in Fig. 2.3. Note that the delayed state model is required only because of the Doppler measurements.

F. Other Parameters for Kalman Filter

1. Transition matrix

The elements of the transition matrix are deduced in a routine manner from the state equation.

They are

$$\phi_{i,i} = 1 \qquad \text{for } i = 1,2,3,4,5,6$$

$$\phi_{i,i} = \cos(\omega.\Delta T) \qquad \text{for } i = 7,8,9,10,11,12$$

$$\phi_{7,8} = \phi_{9,10} = \phi_{11,12} = \frac{\sin(\omega.\Delta T)}{\omega}$$

$$\phi_{8,7} = \phi_{10,9} = \phi_{12,11} = -\ \omega \sin(\omega.\Delta T)$$

$$\phi_{13,13} = \exp(-\beta_{13}.\Delta T)$$

$$\phi_{14,14} = \exp(-\beta_{14}.\Delta T)$$

$$\phi_{15,15} = \exp(-\beta_{15}.\Delta T)$$

$$\phi_{16,16} = 1$$

All other elements are null.

M_n =

$$\begin{bmatrix} a_{An} & b_{An} & c_{An} & 0 & 0 & 0 & d_{An} & 0 & e_{An} & 0 & f_{An} & 0 & -\Delta T & & +\Delta T & 0 \\ 0 & 0 & 0 & a_{Bn} & b_{Bn} & c_{Bn} & d_{Bn} & 0 & e_{Bn} & 0 & f_{Bn} & 0 & 0 & -\Delta T & +\Delta T & 0 \\ \alpha_{An} & \beta_{An} & \gamma_{An} & -\alpha_{Bn} & -\beta_{Bn} & -\gamma_{Bn} & \delta_{An}-\delta_{Bn} & 0 & \varepsilon_{An}-\varepsilon_{Bn} & 0 & \varphi_{An}-\varphi_{Bn} & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Fig. 2.3. Measurement matrices

$$N_n =$$

$$\begin{bmatrix} -a_{An-1} & -b_{An-1} & -c_{An-1} & 0 & 0 & 0 & -d_{An-1} & 0 & -e_{An-1} & 0 & -f_{An-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_{Bn-1} & -b_{Bn-1} & -c_{Bn-1} & -d_{Bn-1} & 0 & -e_{Bn-1} & 0 & -f_{Bn-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Fig. 2.3. (continued)

2. Covariance matrix for white driven states

a. Markov processes: $H_{13,13}$, $H_{14,14}$, $H_{15,15}$ These are given in Brown (1):

$$H_{i,i} = \sigma_i^2 \; [1 - \exp(-2\beta_i \Delta T)] \qquad (2.37)$$

$$H_{i,j} = 0$$

for all $i \neq j$ since we assume the oscillators are independent.

b. Random walk: $H_{16,16}$ From Parzen (11) we have

$$H_{16,16} = \sigma_c^2 \Delta T \qquad (2.38)$$

where σ_c^2 is the parameter of the Wiener process.

3. Uncorrelated measurement error covariance matrix

The measurement error covariance matrix is defined as

$$V = E\left\{ \begin{bmatrix} \delta N_A \\ \delta N_B \\ \delta\tau \end{bmatrix} [\delta N_A \delta N_B \delta\tau] \right\} = [v_{ij}] \qquad (2.39)$$

where $v_{11} = E(\delta N_A^2)$ = variance of uncorrelated count error in receiver A

$v_{22} = E(\delta N_B^2)$ = variance of uncorrelated count error in receiver B

$v_{33} = E(\delta\tau^2)$ = variance of uncorrelated time measurement error

$$v_{12} = v_{21} = E(\delta N_A \delta N_B) = r[E(\delta N_A^2) E(\delta N_B^2)]^{1/2} \qquad (2.40)$$

where r is the crosscorrelation between time uncorrelated count errors in receiver A and receiver B.

Also $v_{31} = v_{32} = v_{13} = v_{23} = 0$

assuming count errors and time errors are not crosscorrelated.

4. Initial estimation error covariance matrix

a. States describing receivers position   Assuming all original estimates of receivers coordinates are not crosscorrelated and about 100 feet r.m.s. we have

$$P_{1,1} = 100^2$$
$$P_{2,2} = P_{1,1}/R_{A_o}^{\ 2}$$
$$P_{3,3} = P_{1,1}/(R_{A_o}\cos\theta_{A_o})^2$$
$$P_{4,4} = 100^2$$
$$P_{5,5} = P_{4,4}/R_{B_o}^{\ 2}$$
$$P_{6,6} = P_{4,4}/(R_{B_o}\cos\theta_{B_o})^2$$
$$P_{i,j} = 0 \text{ for all } i \neq j \qquad \text{for } i = 1,2,3,4,5,6$$
$$j = 1,2,3,4,5,6$$

where R_{A_o} = original estimate of R_A

θ_{A_o} = original estimate of θ_A

and R_{B_o} and θ_{B_o} are the original estimates for R_B and θ_B.

b. States describing satellite position   Assuming 30 feet r.m.s. position error in cross track, along track, and radial satellite coordinates (6) we get

$$P_{7,7} = 30^2$$
$$P_{8,8} = P_{7,7}\omega^2$$
$$P_{9,9} = 30^2/R_{S_o}^{\ 2}$$
$$P_{10,10} = P_{9,9}\omega^2$$
$$P_{11,11} = 30^2/(R_{S_o}\cos\theta_{S_o})^2$$
$$P_{12,12} = P_{11,11}\omega^2$$

where R_{S_0} = original estimate of R_S

θ_{S_0} = original estimate of θ_S

Assuming the original errors on all three coordinates of the satellite are independent and also using relation (2.20)

$P_{i,j} = 0$ for all $i \neq j$ $i = 7,8,9,10,11,12$, $j = 7,8,9,10,11,12$ (satellite states)

c. <u>States describing oscillator and clock errors</u>

$P_{13,13} = E[\delta f_A^2]$ = variance of oscillator correlated error in A

$P_{14,14} = E[\delta f_B^2]$ = variance of oscillator correlated error in B

$P_{15,15} = E[\delta f_S^2]$ = variance of oscillator correlated error in satellite

$P_{16,16} = \sigma_C^2 \cdot T_E$ = variance of synchronization error between clock in receiver A and clock in receiver B and T_E is the time elapsed since the clocks were last synchronized.

Assuming the original estimates of receiver positions, satellite positions, oscillator and clocks synchronization errors are independent:

all other $P_{i,j} = 0$

We now have all the elements to use a Kalman filter.

5. <u>Remarks</u>

The count errors in receivers A and B are partly caused by propagation errors. Therefore one would expect the crosscorrelation between the count error in A and the count error in B to increase as the receivers are brought closer to each other because of the increasing similarity of the two respective propagation paths. The model does not take this into account intrinsically. The crosscorrelation can be changed externally and simulation runs showed that this crosscorrelation does not significantly affect the system performance.

When the system uses two satellite passes it has been assumed it was two different satellites. This is less favorable than using the same satellite twice since then one could have a better estimate of the satellite oscillator error. At the beginning of a second satellite pass all elements of the error covariance matrix corresponding to satellite states (coordinates and oscillator error) are reset to the original value they had at the beginning of the first pass, and their crosscorrelation with other states is reset to zero.

The variance of the clock synchronization error is increased by an amount equivalent to 1 and 1/2 hours of random walk, its crosscorrelation with the states describing the receivers coordinates being maintained the same.

6. Numerical values for error sources in measurements

The numerical values for the sources of error are approximate and claim only to be realistic if not exact.

The satellite oscillator is of crystal type and its offset is assumed to be 25 Hz r.m.s. The receiver local oscillators are assumed to be piloted by the atomic clocks. For a Cesium clock the frequency stability is of the order $\pm 10^{-11}$ for life (8). Then this means a frequency offset of 4×10^{-3} Hz r.m.s for a 400 MHz oscillator. All oscillator offsets are modeled as Markov processes of long time constant compared to the duration of one satellite pass. The time constant is not critical and is set to be 10^8 seconds.

The time synchronization error between both clocks is modeled as a Wiener process of parameter $\sigma = 5 \times 10^{-11}$. This value was deduced from data given in (8).

The Doppler count uncorrelated error is due partly to residual refraction error remaining after correction. From past experience with Doppler navigation satellites (6) we assumed here 10 counts r.m.s due to propagation and 10 counts r.m.s due to other sources. This means about 15 counts r.m.s all together and a crosscorrelation between count error at both receivers of about 0.5 assuming it is due to the propagation errors and that both paths from satellite to receivers are close and have very similar refractions.

The fractional frequency stability of atomic clocks is 10^{-11} r.m.s for averaging times of 1 to 60 seconds. Then the time error introduced in measuring an interval of about 20 seconds is 20×10^{-11} sec r.m.s. This is negligible compared with the time error introduced by the residual refraction errors. The contribution of refraction errors to time error can be arrived at from the 10 counts r.m.s we took for the Doppler error. It corresponds to ten periods of the 400 MHz signal or $0.25\ 10^{-7}$ sec r.m.s. Since we have two receivers and allowing for other sources 5×10^{-8} sec r.m.s of uncorrelated time measurement error seems reasonable.

In summary, the assumed parameters for measurements errors are:

a. Dynamic model

$\beta_A = \beta_B = \beta_S = 10^{-8}\ sec^{-1}$

$\sigma_A = \sigma_B = 4 \times 10^{-3} Hz$

$\sigma_S = 25$ Hz

$\sigma_C = 5 \times 10^{-11}$

b. Measurement model

$E[\delta N_A^2] = E[\delta N_B^2] = 200$

$E[\delta N_A \delta N_B] = 0.5\, E[\delta N_A^2]$

$E[\delta\tau^2] = (5.0 \times 10^{-8})^2$

G. Simplified Translocation System

As it will be explained later in Section III.D, experimental results of simulations indicated it would be worthwhile to investigate a system which would not take into account the satellite position errors. We now give the model for such a system.

The state variables for this simplified system are:

$x_1 = \delta R_A$ ⎫
$x_2 = \delta\theta_A$ ⎬ receiver A
$x_3 = \delta\Lambda_A$ ⎭

$x_4 = \delta R_B$ ⎫
$x_5 = \delta\theta_B$ ⎬ receiver B
$x_6 = \delta\Lambda_B$ ⎭

$x_7 = \delta\Delta_M$ clocks synchronization error

The simplified system also neglects clocks drifts and therefore see all the states to be estimated as biases thus simplifying the computations.

Dynamic model: $x_{n+1} = x_n$ (2.41)

Measurement model: $y_n = M_n x_n + v$ (2.42)

$$M_n = [\alpha_{An}\ \beta_{An}\ \gamma_{An}\ -\alpha_{Bn}\ -\beta_{Bn}\ -\gamma_{Bn}\ -1] \qquad (2.43)$$

The Kalman equations (standard filter) reduce to:

$$b_n = P_{n-1}M_n^T(M_nP_{n-1}M_n^T + V_n)^{-1} \quad (2.44)$$

$$\hat{x}_n = \hat{x}_{n-1} + b_n(y_n - M_n\hat{x}_{n-1}) \quad (2.45)$$

$$P_n = P_{n-1} - b_n(M_nP_{n-1}M_n^T + V_n)b_n^T \quad (2.46)$$

or

$$P_n = (I - b_nM_n)P_{n-1}(I - b_nM_n)^T + b_nVb_n^T \quad (2.47)$$

The actual estimation errors of the simplified system are obtained by considering it to be a suboptimal filter for the full model including satellite position errors and clocks drifts. A simple way to do this in this particular case is shown in Appendix A.

III. SIMULATION RESULTS

A. Introduction

All simulation runs use two passes from different satellites. Each is from north to south and their subtracks are separated by 1300 miles at the equator. The receivers are both in the vicinity of a point halfway between satellites subtracks and 30 degrees latitude north. In each pass the receivers make 36 sets of Doppler and time measurements which are used by the Kalman filter. Since no inertial sensors are used, the accuracy of the system is mainly determined by the relative position of the receivers with respect to satellite (or satellite subtrack) and not function of the position of the receivers on earth.

A first simulation run was made and was used as a reference for comparison with all other runs. In the reference run the two receivers are 50 miles apart, 25 miles east and west of a point halfway between the subtracks. All parameters are as described in Section II.F.6. The crosscorrelation on time uncorrelated Doppler count errors is set to 0.5.

We want to compare three systems:

a) Using Doppler counts only (conventional Transit)

b) Using time measurements only

c) Using both types of measurements

For convenience they will be called Doppler system, Time system and Doppler and Time system respectively.

Fig. 3.1 a,b,c and 3.2 a,b,c show the decay of the estimation errors for all three systems and for altitude, latitude and longitude for two passes (first pass is from first iteration to 36th, and second is pass

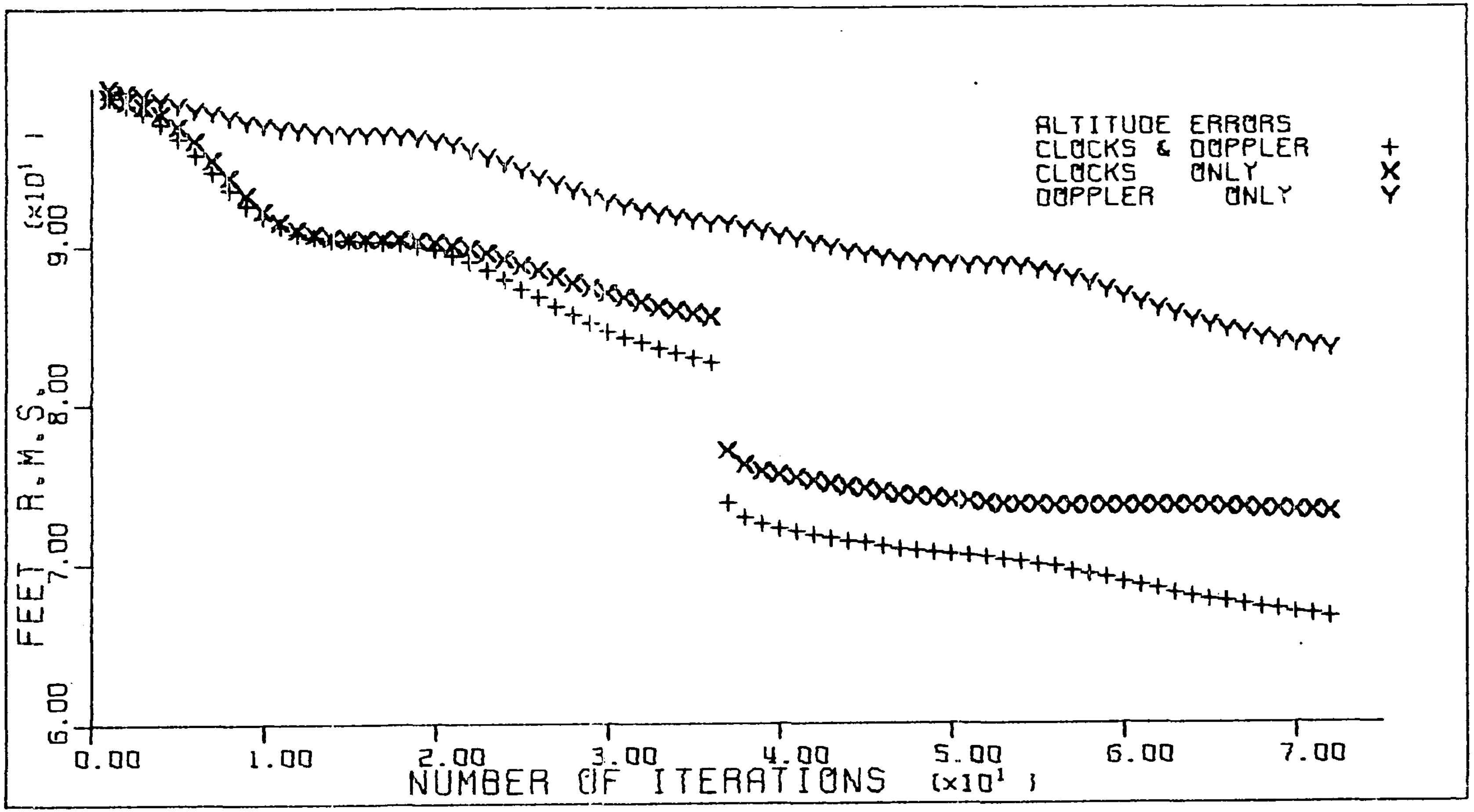

Fig. 3.1a. Absolute position error (altitude)

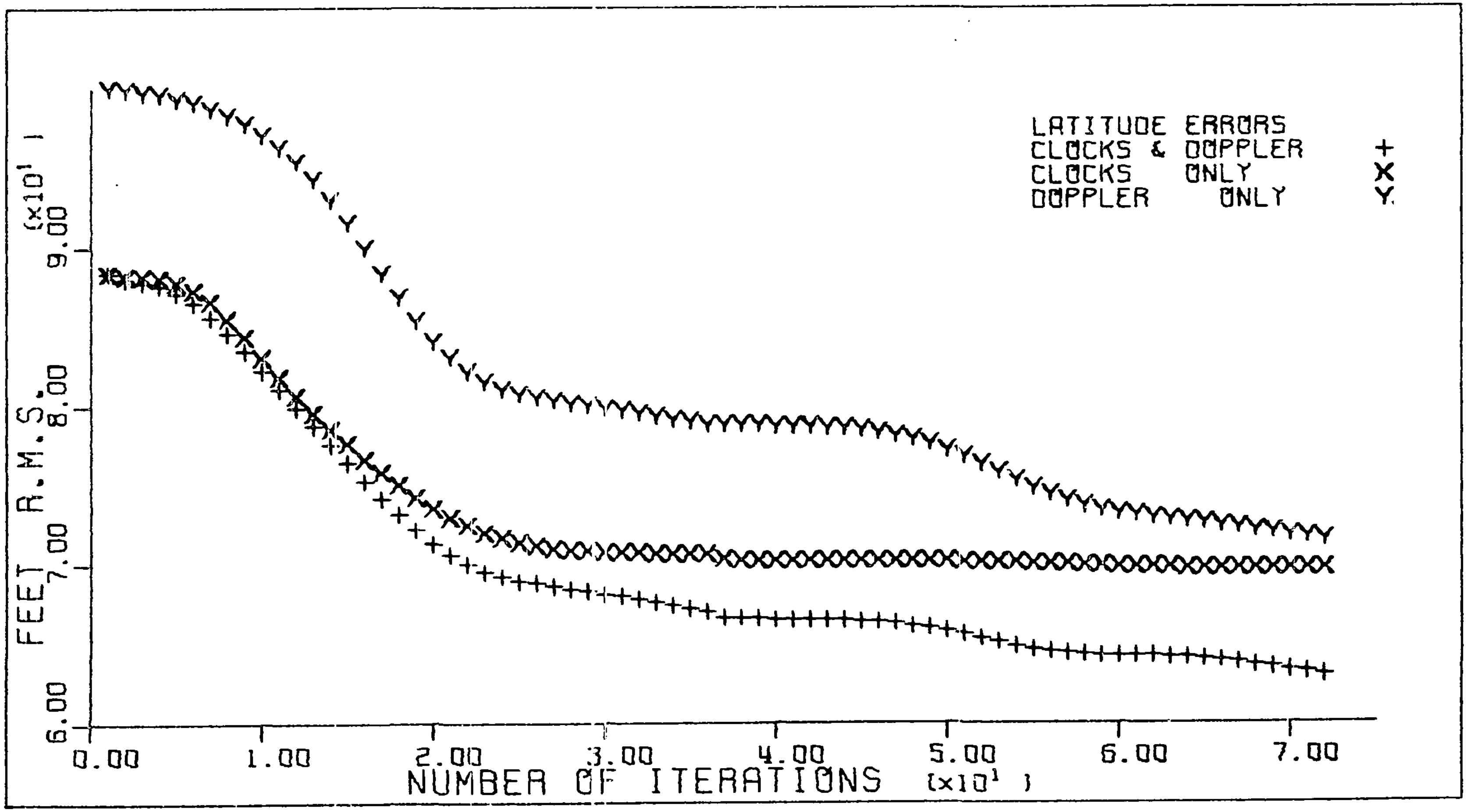

Fig. 3.1b. Absolute position error (latitude)

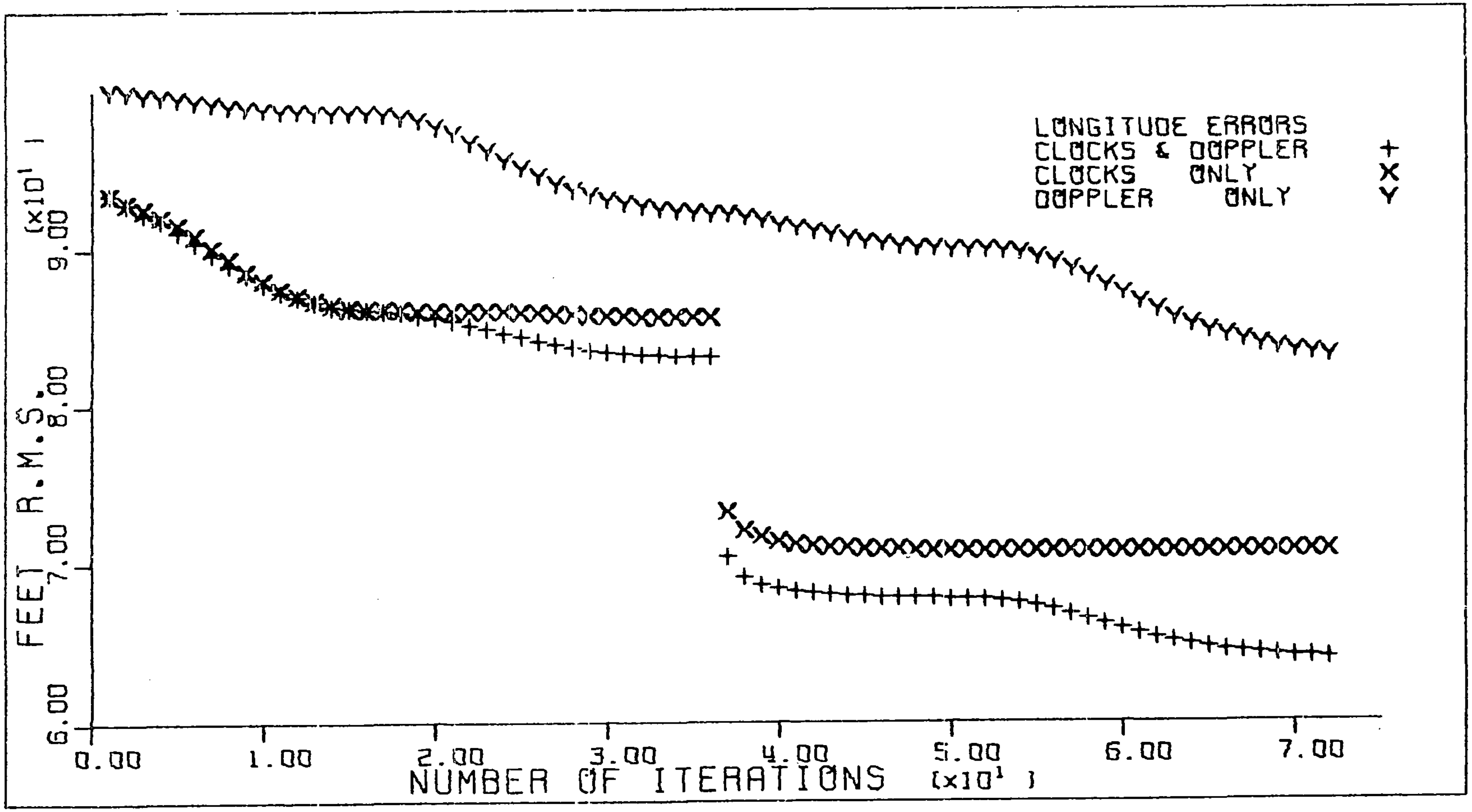

Fig. 3.1c. Absolute position error (longitude)

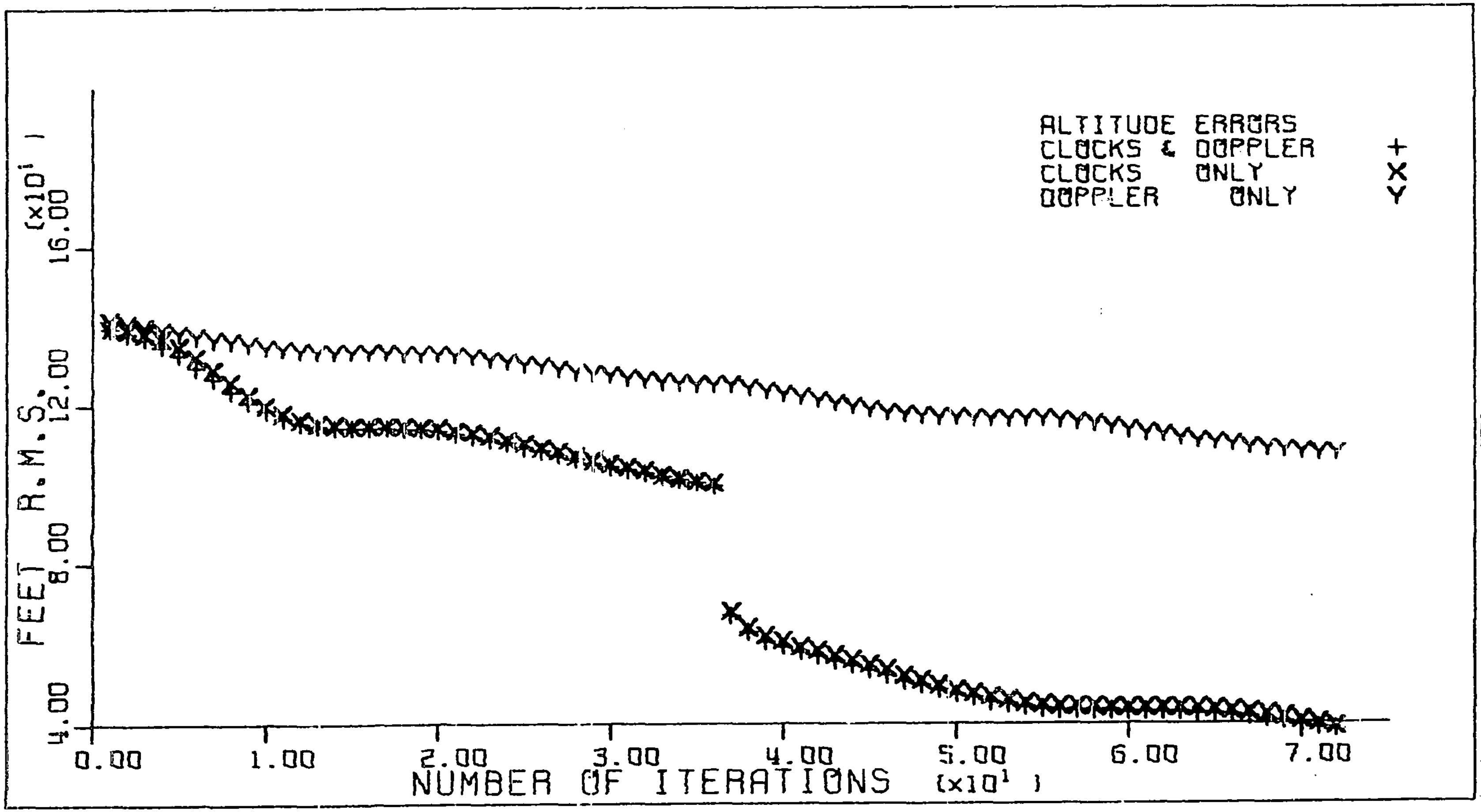

Fig. 3.2a. Relative position error (altitude)

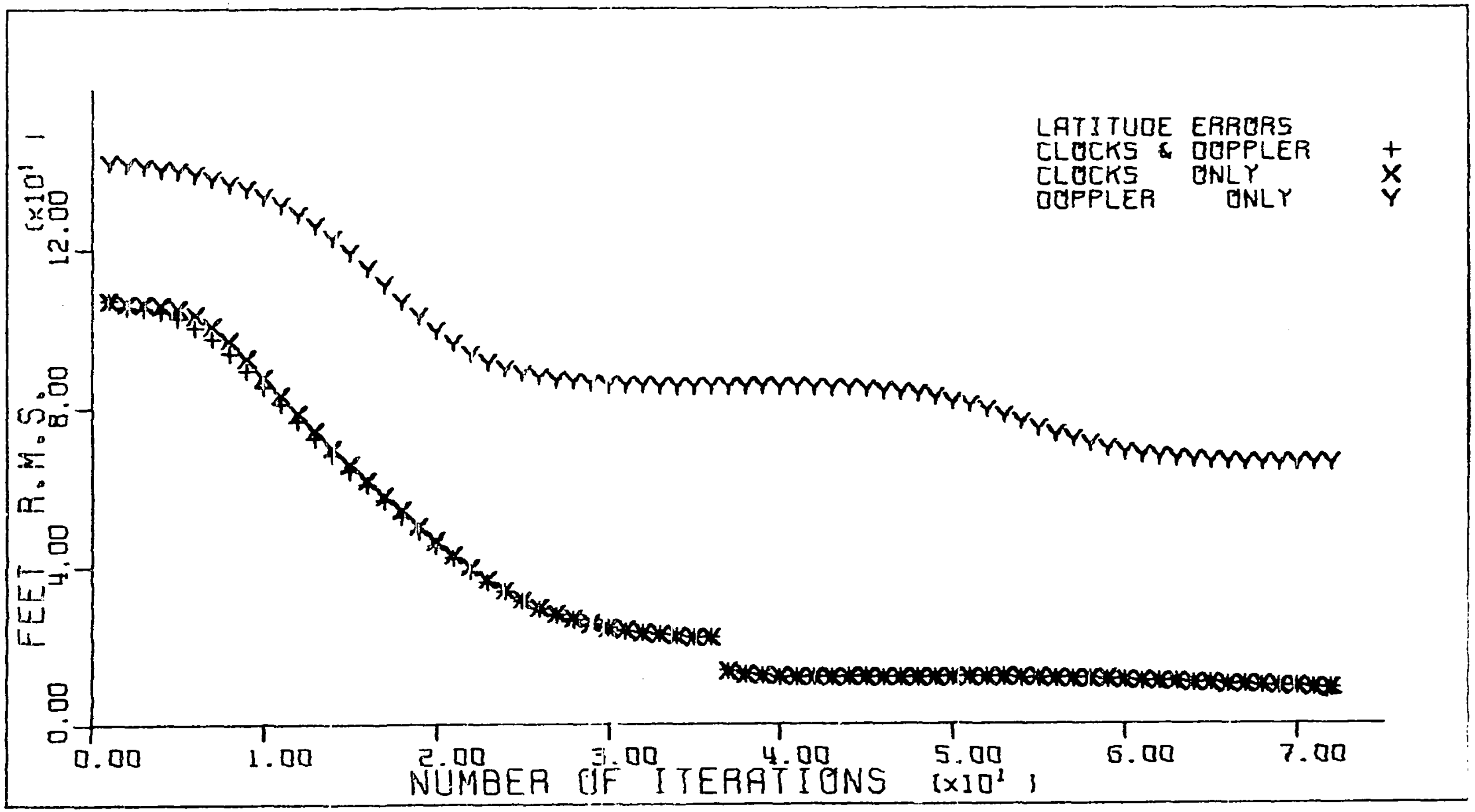

Fig. 3.2b. Relative position error (latitude)

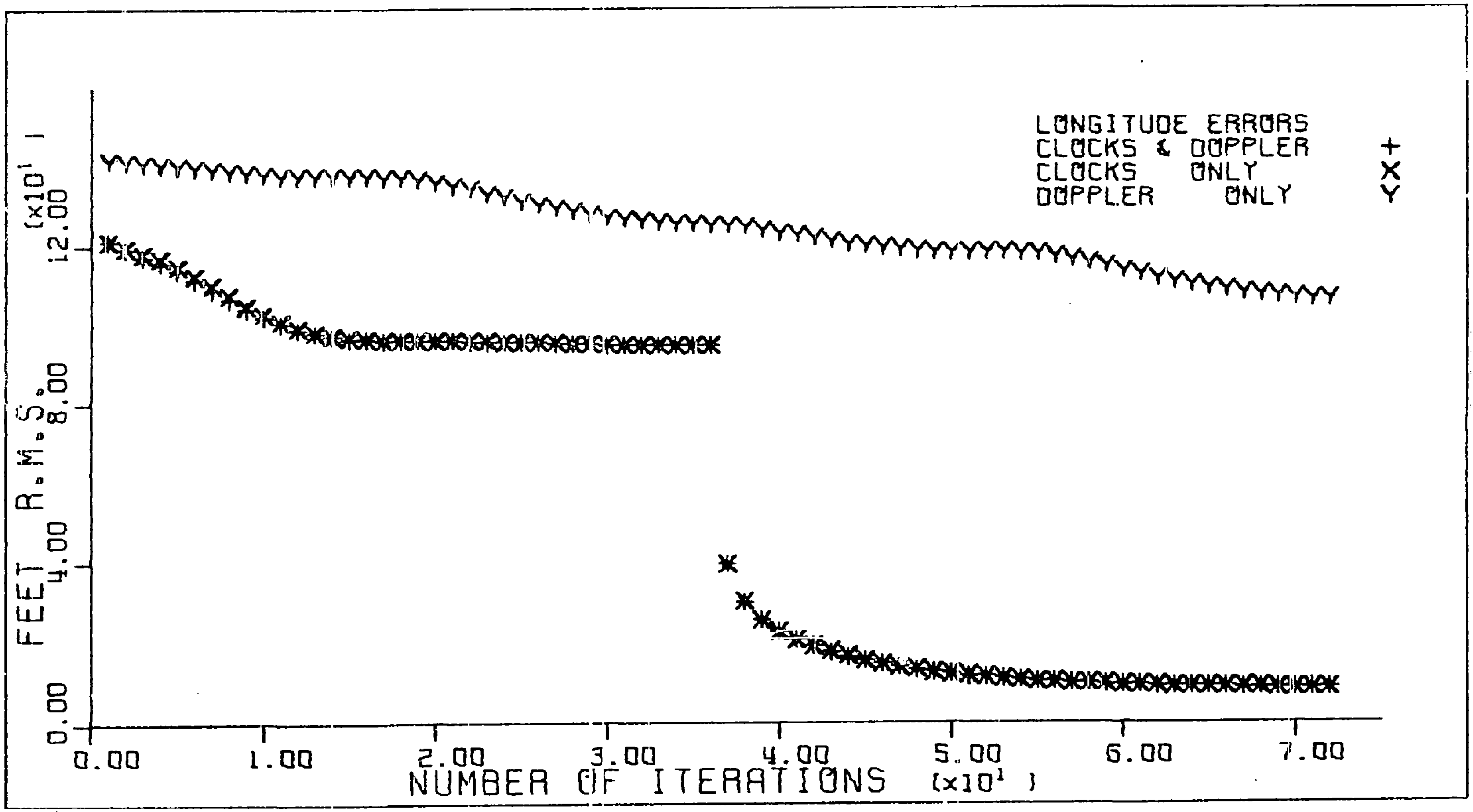

Fig. 3.2c. Relative position error (longitude)

from 37th iteration to 72nd). The r.m.s values of the position errors in altitude, latitude and longitude, for the three systems, at the end of each pass are shown on Table 3.1.

Table 3.1. Expected position errors after one satellite pass and after two satellite passes, for nominal values of parameters (reference run)

Satellite pass	Altitude 1st	Altitude 2nd	Latitude 1st	Latitude 2nd	Longitude 1st	Longitude 2nd
Time and Doppler measurements						
Absolute A	82.7	66.6	67.0	63.0	83.1	64.0
Absolute B	84.9	66.7	68.4	63.0	81.8	64.1
Relative	99.8	38.3	21.9	9.2	94.7	8.9
Time measurements only						
Absolute A	85.6	73.2	70.7	69.8	85.6	70.8
Absolute B	87.7	73.3	72.3	69.8	84.5	70.8
Relative	100.2	39.3	22.0	9.2	94.7	8.9
Doppler measurements only						
Absolute A	91.4	83.5	78.9	71.6	92.2	83.1
Absolute B	92.4	83.5	79.9	71.6	92.0	83.1
Relative	125.1	107.4	85.6	65.8	125.3	106.2

This table shows that the Time system does a little better than the Doppler system for absolute positioning, and the Time system is much better than the Doppler system for relative positioning.

The above demonstrates that when using both types of measurements there is some improvement for absolute positioning, while for relative positioning the time measurements give much better results and make the Doppler measurements worthless.

B. Influence of Crosscorrelation

Two runs like the reference run were made where the crosscorrelation in Doppler counts was changed to 0.0 and 0.9. The results are shown in Table 3.2, and they show that the conclusions made from the reference run remain valid. The Time system is not dependent on this crosscorrelation so it is not shown in these tables. The relative positioning accuracy of the Doppler system improves as the crosscorrelation increases.

Table 3.2. Influence of the crosscorrelation between the Doppler count errors in both receivers

	Altitude		Latitude		Longitude	
Satellite pass	1st	2nd	1st	2nd	1st	2nd
			Crosscorrelation = 0.0			
Time and Doppler measurements						
Absolute A	81.9	64.5	65.4	60.4	82.3	61.7
Absolute B	84.0	64.5	66.8	60.4	80.9	61.7
Relative	100.0	38.8	21.9	9.2	94.7	8.8
Doppler measurements only						
Absolute A	92.8	86.3	82.7	74.1	93.5	86.0
Absolute B	93.7	86.3	83.6	74.1	93.3	86.0
Relative	131.3	121.0	103.1	84.3	131.2	120.1
			Crosscorrelation = 0.9			
Time and Doppler measurements						
Absolute A	82.7	67.2	67.6	64.1	83.4	65.2
Absolute B	84.9	67.3	69.1	64.1	82.2	65.2
Relative	98.3	34.7	21.7	8.9	94.4	8.8
Doppler measurements only						
Absolute A	86.1	72.9	71.0	66.8	87.8	72.6
Absolute B	87.8	72.9	72.5	66.8	87.4	72.6
Relative	108.7	65.5	48.3	32.7	110.2	64.3

Table 3.3. Influence of the distance between receivers (receivers 5 miles apart

Satellite pass	Altitude 1st	Altitude 2nd	Latitude 1st	Latitude 2nd	Longitude 1st	Longitude 2nd
Time and Doppler measurements						
Absolute A	83.7	66.8	67.7	63.8	82.5	64.1
Absolute B	83.9	66.8	67.8	63.8	82.4	64.1
Relative	99.8	38.2	21.8	9.1	94.6	8.5
Time measurements only						
Absolute A	86.6	73.4	71.5	70.9	85.1	70.8
Absolute B	86.8	73.4	71.6	70.8	85.0	70.8
Relative	100.3	39.3	21.9	9.2	94.6	8.5
Doppler measurements only						
Absolute A	91.9	83.6	79.4	71.6	92.1	83.1
Absolute B	92.0	83.6	79.5	71.6	92.1	83.1
Relative	125.1	107.5	85.6	65.8	125.2	106.2

Table 3.4. Receivers north-south of each other

Satellite pass	Altitude 1st	Altitude 2nd	Latitude 1st	Latitude 2nd	Longitude 1st	Longitude 2nd
Time and Doppler measurements						
Absolute A	83.7	66.6	67.7	63.5	82.3	62.8
Absolute B	84.0	67.0	67.6	63.7	82.5	63.3
Relative	100.0	39.1	22.0	9.3	94.2	8.5
Time measurements only						
Absolute A	86.5	73.3	71.5	70.5	84.9	69.2
Absolute B	86.8	73.6	71.4	70.8	85.2	69.7
Relative	100.4	40.3	22.1	9.5	94.3	8.5
Doppler measurements only						
Absolute A	91.9	83.5	79.4	71.5	92.0	83.0
Absolute B	92.0	83.7	79.4	71.6	92.2	83.2
Relative	125.1	107.5	85.7	65.9	124.7	105.8

Tables 3.3 and 3.4 show that the results are essentially the same if the two receivers are closer to each other or spread in the north-south direction instead of east-west. Therefore independently of the receivers' relative positions, for a distance of the order of 50 miles between receivers, the conclusions remain the same as for the reference run. The Time system is a little better for absolute positioning but not very significantly considering the lack of accuracy on fixing the parameters of each error source in the Doppler measurements and time measurements. For relative positioning the Time system is much better than the Doppler system. Using both time measurements and Doppler measurements is equivalent to the Time system for relative positioning and a little better for absolute positioning.

C. Satellite Pass Geometry and Estimation of Position

The satellites are assumed to be in polar circular orbits. In practice they are only in near circular orbits but this approximation does not change significantly the bearing of satellite position errors on the estimates of the receivers position errors.

The pass geometry is related to the receiver position estimate errors and also to the relative magnitude of these errors in altitude, latitude and longitude.

From the linearization equation (2.24) one can consider the part of range variation due to variation of the receiver coordinates alone. Or:

$$\delta\rho = A\delta R + B\delta\theta + C\delta\Lambda$$

This can be rewritten in terms of variations in feet in vertical, east-west, and north-south directions using the local coordinate system for the receiver.

$$\delta\rho = A'\delta z + B'\delta x + C'\delta y$$

These coefficients give directly the variation of the range in feet caused by variations in either direction in feet and are plotted for the two passes in Fig. 3.3. These express how sensitive the range is to variations of receiver coordinates in any of the three directions.

The decay of the estimation errors in feet in all three directions for conventional receivers using Doppler counts alone, for receivers using time measurements and receivers using both is shown in Fig. 3.1 a,b,c for absolute position of one receiver and Fig. 3.2 a,b,c for the relative position of one receiver with respect to the other. These curves show that after one pass (36th iteration) the latitude error is much smaller than the altitude or longitude errors. The plot of the linearization coefficients shows that the coefficients corresponding to altitude and longitude are of comparable magnitude and vary in a similar fashion during the first pass. Then the estimator cannot separate one from the other, and it gives a poor estimate for both. With the second pass on the opposite side of the receivers (37th to 72nd iterations), the longitude coefficient changes sign. Then for both passes together all three coefficients behave differently enough to enable the filter to separate the errors in all three directions. This illustrates the fact that the distribution of the uncertainty in position between the three directions is mainly a question of the geometry of the satellites passes and that some insight into it can be gained by directly looking at the linearization equation used in the modeling. This also implies that if

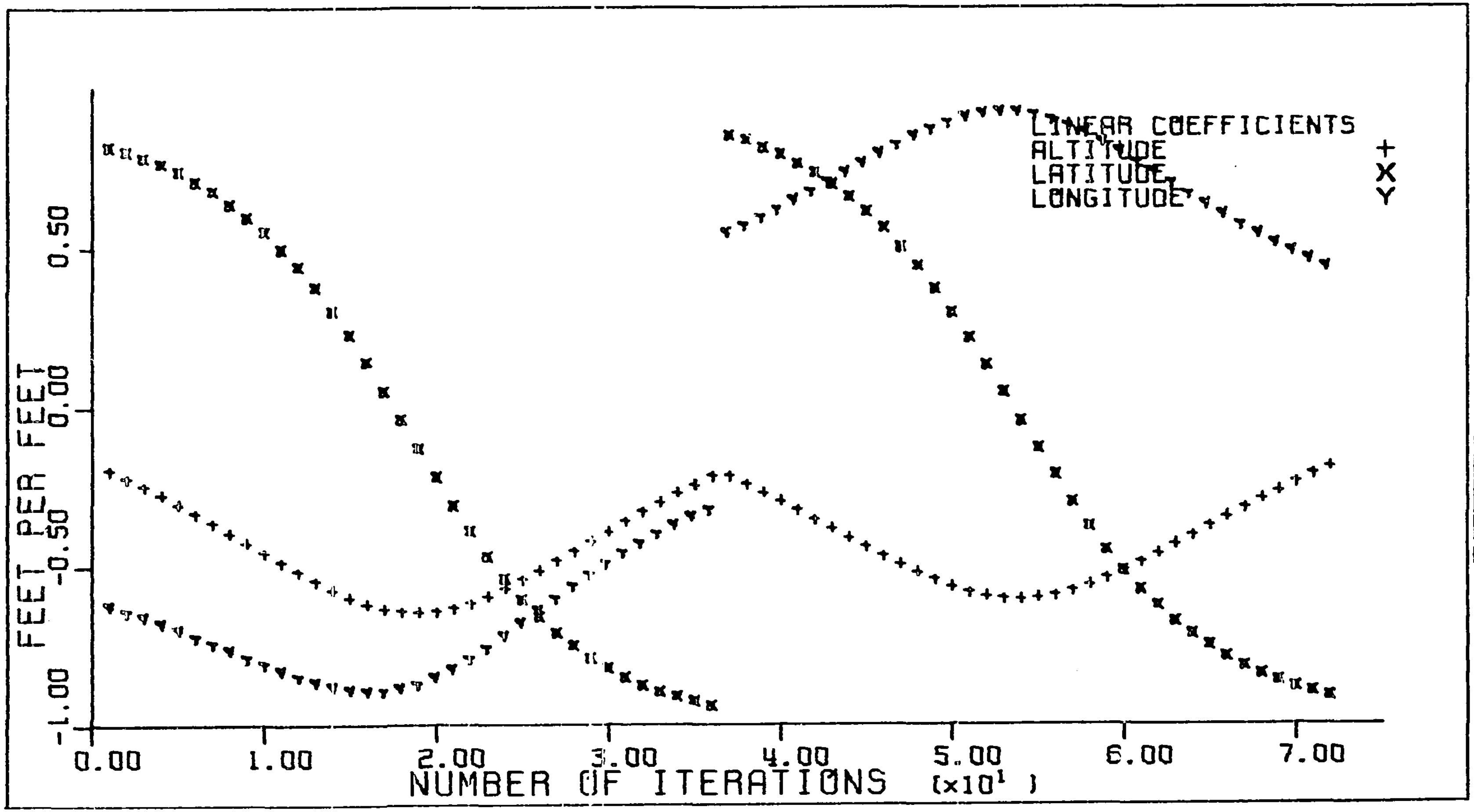

Fig. 3.3. Normalized linearization coefficients for two passes

the altitude is initially known accurately then the system will give a better estimate of the longitude, and vice versa, a good initial estimate of the longitude enables the system to give a better estimate of the altitude.

D. Simplified System

It has been noticed that using time measurements alone gives good results for relative positioning. As part of its operation the Kalman filter estimates the satellite position errors, but the improvement it makes on their original estimates is very small. The decay of the variances corresponding to states describing satellite position errors is less than one per cent in one satellite pass. Therefore a simplified system using time measurements alone but which would not estimate the satellite position errors should perform about as well. The model for such a system was given before in Section II.G, and, like the Time system, it does not require a delayed state Kalman filter. Also, the system state vector is reduced from thirteen to seven elements which yields considerable simplification. The recursive equations are further simplified because the dynamic model is trivial involving only states which do not vary with time.

Fig. 3.4a and b show the performance of the simplified system for circumstances identical to those of the reference run. Comparing these plots with those for the Time system shows that there is no appreciable loss of accuracy in either relative or absolute positioning.

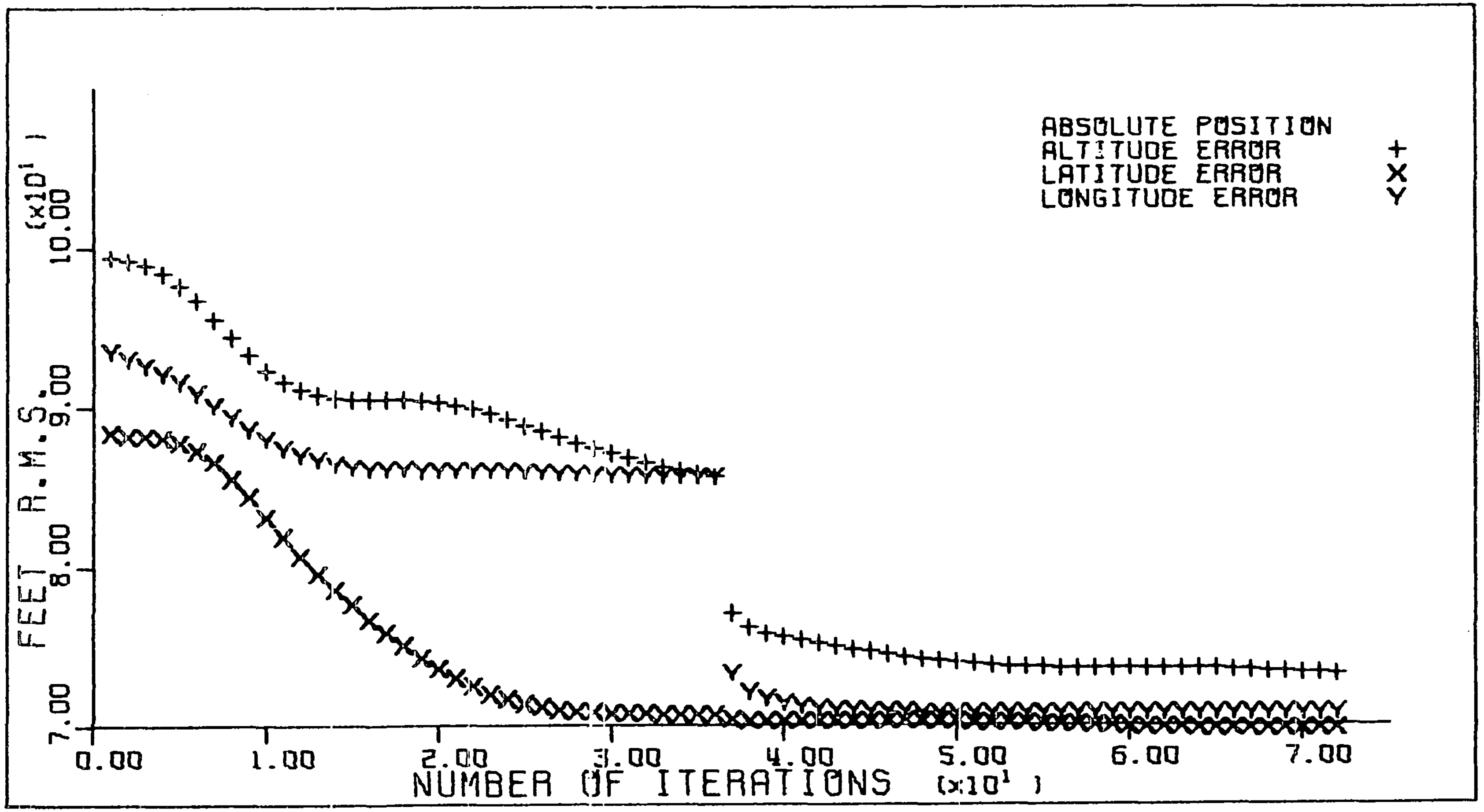

Fig. 3.4a. Simplified system (absolute position errors)

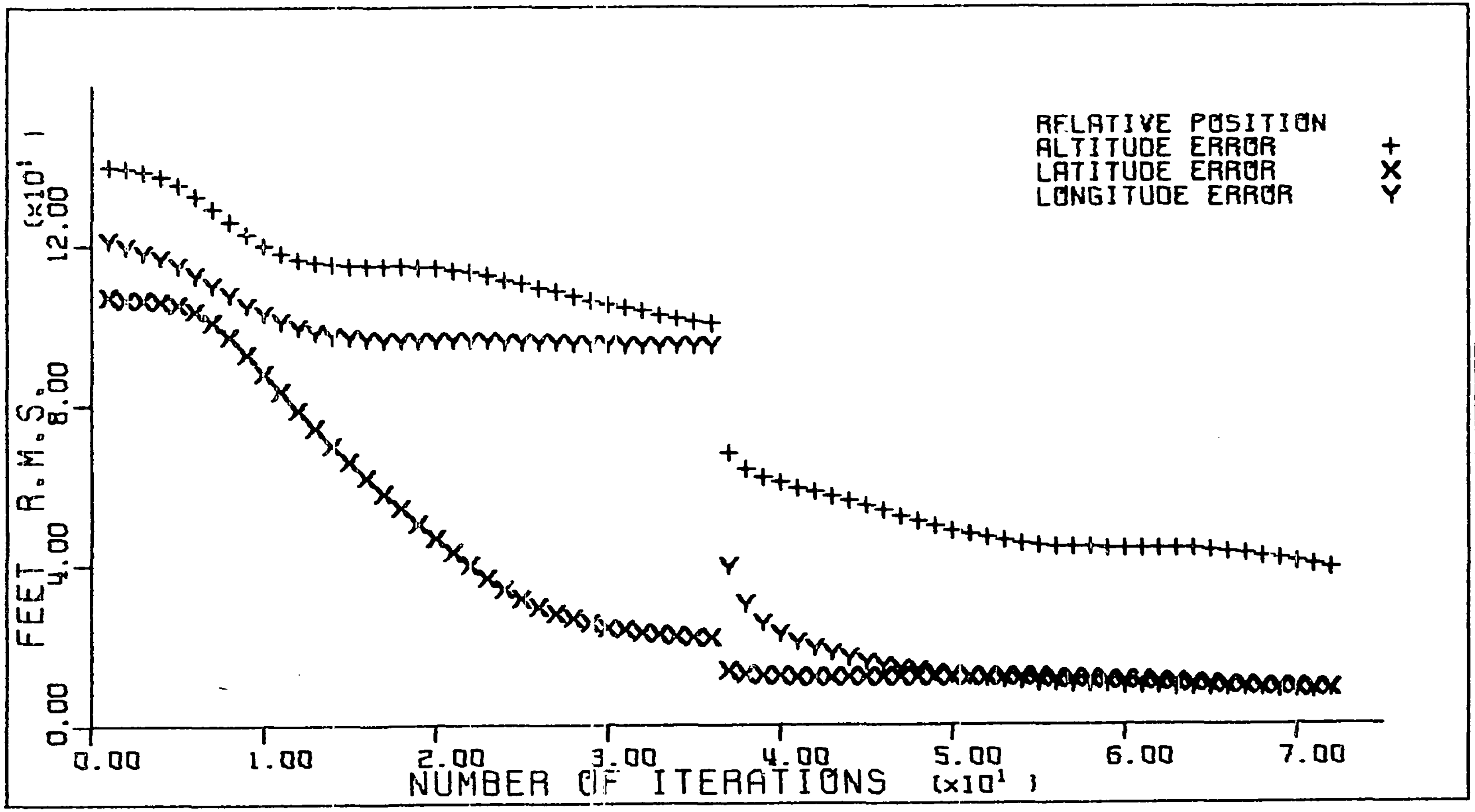

Fig. 3.4b. Simplified system (relative position errors)

E. Clocks Synchronization Error

One interesting aspect of the Time system or the Simplified system is that it is not necessary to have a good synchronization between the clocks in both receivers.

A simulation run was made where the original expected synchronization error was very high, 10^9 times the value used in the reference run, and the performances of both the Time system and the Simplified system were not significantly affected. This is because during a satellite pass the synchronization error is practically a constant which is easily estimated by the Kalman filter and accounted for in the estimation of positions. This results in an apparent "self alignment" of the clocks which suppresses the synchronization problem all together.

F. Satellite Position Error

It has been mentioned that when using two receivers for relative positioning, satellite position errors tend to cancel out and have little bearing on the relative position error. Also, when using time measurements one can expect little influence of satellite position error on relative position error of the receivers since the time lag measured is much more sensitive to relative motions of a receiver with respect to the other than it is to comparable motions of the satellite. This is checked by a simulation run where satellite position errors were raised to 300 feet r.m.s for each coordinate instead of the 30 feet r.m.s used in the reference run. The accuracy is slightly reduced, more so for the simplified system than for the Time system as shown by Table 3.5. More surprisingly this table shows that the absolute position estimates are also practically unaffected by the satellite position error.

Table 3.5. Position errors for Time system and Simplified system for high original uncertainty on satellite position

Satellite pass	Altitude 1st	Altitude 2nd	Latitude 1st	Latitude 2nd	Longitude 1st	Longitude 2nd
Time system						
Absolute A	86.4	73.4	70.9	70.1	87.8	70.9
Absolute B	88.3	73.4	72.4	70.1	87.7	70.9
Relative	102.7	40.6	24.2	9.69	103.9	12.2
Simplified system						
Absolute A	86.4	73.4	70.9	70.3	87.8	71.0
Absolute B	88.3	73.4	72.4	70.3	87.6	71.0
Relative	102.7	40.6	24.2	9.73	103.9	13.5

IV. CONCLUSION

The goal of this study was to find out how much improvement could be expected, when using time measurements in addition to the Doppler measurements normally found in Transit systems, when two receivers are used for geodesy.

Simulation runs indicate that there should be a great improvement in accuracy both for absolute and relative positioning. In the case of relative positioning, Doppler data could be left out entirely since simulation indicates that using time measurements alone gives nearly as good results as using both time and Doppler measurements. This would simplify the receivers and the associated data processing.

When using time measurements only, simulation shows that neglecting the satellite position errors in the filtering process does not significantly affect performance. This could further simplify the software part of the system.

A striking result is that accurate synchronization of the clocks is not necessary.

Then the main difference in the implementation of a system using time measurements compared to one using Doppler measurements is the extra two clocks. Precision Cesium clocks are relatively expensive but might well be feasible in many surveying applications.

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VII. APPENDIX A:

SIMULATION OF SIMPLIFIED SYSTEM

The model for the simplified system is given by equations (2.41, 42, 43) and its filter algorithm by equations (2.44 to 2.47).

Since we need the full model to get the actual errors of the simplified system it is simpler to simulate the simple system using the program for the full system. Consider the partitioned system:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{dynamic model} \tag{A.1}$$

$$y = (M_1 M_2) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + v \quad \text{measurement model} \tag{A.2}$$

Let $P_{n-1} = \begin{bmatrix} P_{n-1} & 0 \\ 0 & 0 \end{bmatrix}$ and $H = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Then using Kalman equations one gets:

$$b_n = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}_n = \begin{bmatrix} p_{n-1} M_1^T (M_1 p_{n-1} M_1^T + v)^{-1} \\ 0 \end{bmatrix} \tag{A.3}$$

$$P_n = \begin{bmatrix} (I - b_1 M_1) p_{n-1} (I - b_1 M_1)^T + b_1 V b_1^T & 0 \\ 0 & 0 \end{bmatrix} \tag{A.4}$$

b_1 is the same as given by equation (2.44) and the upper left corner of P_n is the same as given by (2.47). Therefore the above behaves like the simplified system using the same Kalman equations (including full measurement equations) as the full system, the only difference being the initial P and H matrices. Then we can use two sets of error covariance matrices, one for the error seen by the simplified system and one for the actual errors and cycle them through the same Kalman recursive equations in the following manner:

1) Compute the suboptimal gain b_n from error covariance matrix seen by the simplified system using (2.44).

2) Update the error covariance matrix seen by the simplified system using (2.47).

3) Update actual error covariance matrix for full system using (2.11) and compute actual estimates of position errors of simplified system.

VIII. APPENDIX B:

ROUND-OFF ERRORS

The recursive equation for the error covariance matrix is $P_n = (I - b_nM_n)P_{n-1}(I - b_nM_n)^T + b_nVb_n^T$. One way of computing is as follows:

compute: b_nM_n

then: $(I - b_nM_n)$

then: $(I - b_nM_n)P_{n-1}(I - b_nM_n)^T + b_nVb_n^T$

This causes round-off errors to make the covariance matrix very unsymetric and make the simulation invalid. This happens because elements of P_{n-1} are much greater than elements of $b_nM_nP_{n-1}$ with which they are added in both pre and post multiplications. Separating smaller and bigger terms alleviates this. We rewrite:

$$P_n = P_{n-1} - b_nM_nP_{n-1} - P_{n-1}M_n^Tb_n^T + b_nM_nP_{n-1}M_n^Tb_n^T + b_nVb_n^T$$

The above products are computed before summing and nonsymmetry is generated by the fourth term alone. So doing the relative difference between corresponding off-diagonal terms in one step of computation is less than 0.01%, compared to more than 100% using the first method, before it is symetrized by doing:

$$\text{new } P_{ij} = \frac{P_{ij} + P_{ji}}{2}$$

IX. APPENDIX C:

COMPUTER PROGRAM LISTING FOR REFERENCE RUN

```
T/
C VARIANCE ANALYSIS OF SURVEYING SYSTEM USING SATELLITES
      IMPLICIT  REAL*8(A-H,O-Z),INTEGER(I-N)
      INTEGER DOP,CLOC
      REAL*8  MN1,MN2,NN1,MB,MPH,MH,LAMBDI
      DIMENSION P(16,16),PC(16,16),PD(16,16)
      COMMON PCL(3,3),A(3,3,3)
     1),NN1(2,16),MN2(1,16),AC(6),LDUM
      COMMON /KAL/VN1(2,2),H(16,16),VN2,PHI( 16,16),MN1(2,16
C
C.........................NON TIME VARYING ELEMENTS........
C NUMBER OF ITERATIONS IN ONE SATELLITE PASS
      NUMIT=36
C  ENTER CONSTANTS
      DT=20.0D0
      RE=2.092574 D07
      GM=0.1407654D17
      PCL(1,3)=RE+600.0D0*6080.0D0
      OMEGS=DSQRT(GM/PCL(1,3)**3)
      OMEGA=7.292115D-5
      PI=3.14159265358979 3D0
      FEET=6080.0D0
      RAD=PI/180.0D0
      CELI=0.3048D0/3.0D8
      LAMBDI=400.0D6*CELI
C  ORIGINAL GROUND STATIONS COORDINATES ESTIMATES
      TETAO=30.0D0*RAD
      POL(1,1)=2.092574D07
      POL(2,1)=TETAO+25.0D0*FEET/POL(1,1)
      POL(3,1)=0.0D0
      POL(1,2)=2.092574D07
      POL(2,2)=TETAO-25.0D0*FEET/POL(1,1)
      POL(3,2)= 0.0D0
      B13=1.0D-8
      B14=B13
      B15=B13
C CORRELATED ERROR VARIANCES
      VM13= 4.0D-3**2
      VM14=VM13
      VM15=25.0D0**2
      VM16=5.0D-11**2
C UNCORRELATED ERROR VARIANCES
      VW13=2.0D2
      VW14=VW13
      VW16=5.0D-8**2
C INITIALIZE MATRICES
      DO 105 I=1,16
      DO 105 J=1,16
      P(I,J)=0.0D0
      PHI(I,J)=0.0D0
```

```
      H(I,J)=C.ODO
      IF(I.GT.1) GO TO 103
      MN2(I,J)=0.0D0
  103 IF(I.GT.2) GO TO 105
      MN1(I,J)=C.0D0
      NN1(I,J)=C.0D0
  105 CONTINUE
C COMPUTE WHITE DRIVEN STATES COVARIANCE MATRIX
      H(13,13)=VM13*(1.0D0-DEXP(-2.0D0*B13*DT))
      H(14,14)=H(13,13)
      H(15,15)=VM15*(1.0D0-DEXP(-2.0D0*B15*DT))
      H(16,16)= VM16*DT
C COMPUTE TRANSITION MATRIX PHI
      DO 110 I=1,6
  110 PHI(I,I)=1.0D0
      DO 112 I=7,12
  112 PHI(I,I)= DCOS(OMEGS*DT)
      DO 114 I=7,11,2
  114 PHI(I,I+1)= DSIN(OMEGS*DT)/OMEGS
      DO 116 I=8,12,2
  116 PHI(I,I-1)= -DSIN(OMEGS*DT)*OMEGS
      PHI(13,13)=  DEXP(-B13*DT)
      PHI(14,14)=PHI(13,13)
      PHI(15,15)=PHI(13,13)
      PHI(16,16)= 1.0D0
C INITIALIZE ERROR COVARIANCE MATRIX
      P(1,1)=1.0D4
      P(2,2)=P(1,1)/RE**2
      P(3,3)=P(1,1)/(RE*DCOS(PCL(2,1)))**2
      P(4,4)=1.0D4
      P(5,5)=P(4,4)/RE**2
      P(6,6)=P(4,4)/(RE*DCOS(POL(2,2)))**2
      P(7,7)=30.0D0**2
      P(8,8)=OMEGS*OMEGS*P(7,7)
      P(9,9)=(30.0D0/POL(1,3))**2
      P(10,10)=OMEGS*OMEGS*P(9,9)
      P(11,11)=(30.0D0/(POL(1,3)*DCOS(TETAO-OMEGS*NUMIT*10.
     1ODO)))**2
      P(12,12)=P(11,11)*OMEGS*OMEGS
      P(13,13)=VM13
      P(14,14)=VM14
      P(15,15)=VM15
      P(16,16)=  VM16*36.0D2*24*30
C PC COVARIANCE MATRIX CLOCKS ALONE, PD DOPPLER ALONE
      DO 130 J=1,16
      DC 130 I=1,16
      PD(I,J)=P(I,J)
  130 PC(I,J)=P(I,J)
C COMPUTE MEASUREMENTS ERROR  COVARIANCE MATRICES
      COR=0.5DC
```

```
      VN1(1,1)=VW13
      VN1(1,2)=COR*VW13
      VN1(2,1)=VN1(1,2)
      VN1(2,2)=VW14
      VN2=VW16
C
C  ...........TIME DEPENDENT  COMPUTATIONS.................
      SLAM= .196D0
      NPASS=0
    8 NPASS=NPASS+1
      LDUM=0
      T=-NUMIT*10.0D0
   10 T=T+DT
      LDUM=LDUM+1
C  SATELLITE COORDINATES
      POL(2,3)=CMEGS*T+TETAO
      POL(3,3)=-CMEGA*T+SLAM
C  COMPUTE MEASUREMENTS MATRICES
C STORE PART CF OF OLD MN1 AS NEW NN1
      DO 210  I=1,2
      DO 210  J=1,11
  210 NN1(I,J)=-MN1(I,J)
C COMPUTE NEW MEASUREMENTS MATRICES
  215 CALL  CCEF
      MN1(1,1)=A(1,1,3)*LAMBDI
      MN1(1,2)=A(2,1,3)*LAMBDI
      MN1(1,3)=A(3,1,3)*LAMBDI
      MN1(1,7)=A(1,3,1)*LAMBDI
      MN1(1,9)=A(2,3,1)*LAMBDI
      MN1(1,11)=A(3,3,1)*LAMBDI
      MN1(1,13)=-DT
      MN1(1,15)=DT
      MN1(2,4)=A(1,2,3)*LAMBDI
      MN1(2,5)=A(2,2,3)*LAMBDI
      MN1(2,6)=A(3,2,3)*LAMBDI
      MN1(2,7)=A(1,3,2)*LAMBDI
      MN1(2,9)=A(2,3,2)*LAMBDI
      MN1(2,11)=A(3,3,2)*LAMBDI
      MN1(2,14)=-DT
      MN1(2,15)=DT
      MN2(1,1)=A(1,1,3)*CELI
      MN2(1,2)=A(2,1,3)*CELI
      MN2(1,3)=A(3,1,3)*CELI
      MN2(1,4)=-A(1,2,3)*CELI
      MN2(1,5)=-A(2,2,3)*CELI
      MN2(1,6)=-A(3,2,3)*CELI
      MN2(1,7)=(A(1,3,1)-A(1,3,2))*CELI
      MN2(1,9)=(A(2,3,1)-A(2,3,2))*CELI
      MN2(1,11)=(A(3,3,1)-A(3,3,2))*CELI
      MN2(1,16)=-1.0D0
```

```
C  LINEAR COEFS FEET TO FEET
      AC(1)=A(1,1,3)
      AC(2)=A(2,1,3)/RE
      AC(3)=A(3,1,3)/RE/DCOS(POL(2,1))
      AC(4)=A(1,2,3)
      AC(5)=A(2,2,3)/RE
      AC(6)=A(3,2,3)/RE/DCOS(POL(2,2))
      WRITE(6,1215)(AC(I),I=1,6)
 1215 FORMAT(' ',T13,6D15.4)
C     COMPUTE NEW COVARIANCE MATRICES
      CALL KALMAN(1,1, P)
      CALL KALMAN(0,1,PC)
      CALL KALMAN(1,0,PD)
  900 IF(LDUM.LT.NUMIT) GO TO 10
      IF (NPASS.EQ.2) GO TO 500
C
C REINITIALIZE PART OF COVARIANCE MATRIX FOR NEW  SATELLITE
      DO 380 J=1,16
      DO 380 I=1,J
      IF(J.LE.6) GO TO 380
      IF(J.GE.7.AND.J.LE.12.OR.J.EQ.15)GO TO 375
      IF(I.LE.6) GO TO 380
      IF(I.EQ.13.OR.I.EQ.14.OR.I.EQ.16)GO TO 380
  375 P(I,J)= C.0D0
      PC(I,J)=C.0D0
      PC(I,J)=C.0D0
  380 CONTINUE
      P(7,7)=30.0D0**2
      P(8,8)=CMEGS*CMEGS*P(7,7)
      P(9,9)=(30.0D0/POL(1,3))**2
      P(10,10)=CMEGS*CMEGS*P(9,9)
      P(11,11)=(30.0D0/(POL(1,3)*DCOS(TETAO-OMEGS*NUMIT*10.
     1ODO)))**2
      P(12,12)=P(11,11)*OMEGS*CMEGS
      P(15,15)=VM15
      P (16,16)=P (16,16)+VM16*36.0D2*2.0D0
      PC(16,16)=PC(16,16)+VM16*36.0D2*2.0D0
      PD(16,16)=PD(16,16)+VM16*36.0D2*2.0D0
      DO 390 I=7,15
      IF(I.EQ.13) GO TO 390
      IF(I.EQ.14) GO TO 390
      PD(I,I)=P(I,I)
      PC(I,I)=P(I,I)
  390 CONTINUE
      DO 395 I=1,16
      DO 395 J=1,I
      P(I,J)=P(J,I)
      PC(I,J)=PC(J,I)
      PC(I,J)=PC(J,I)
  395 CONTINUE
```

```
      SLAM=-.196D0
      GO TO 8
  500 STOP
      END
C
C
C
C
C
C
      SUBROUTINE KALMAN(DOP,CLOC,P)
C  THIS SUBROUTINE DOES ONE STEP OF KALMAN ALGORYTHM
      IMPLICIT  REAL*8(A-H,O-Z),INTEGER(I-N)
      INTEGER DOP,CLOC
      REAL*8  MN1,MN2,NN1,MB,MPH,MH
      DIMENSION COV(9),MB(2,16),MPH(2,16),PHIT(16,16),
     1DUM1( 2,16),DUM2( 2,16),QP(2,2),MH(2,2),QN(2,2),
     2DUM3(16, 2),STA(9)  ,QNI(2,2) ,
     3P(16,16),BQ(16,2) ,BN1(16,2),PHIP(16,16),PPPT(16,16)
      COMMON PCL(3,3),A(3,3,3)
      COMMON  /MAT/N,JN,JM
      COMMON /KAL/VN1(2,2),H(16,16),VN2,PHI( 16,16),MN1(2,16
     1),NN1(2,16),MN2(1,16),AC(6),LDUM
      EQUIVALENCE(MPH,DUM1,DUM3,BQ ),(MB,BN1),(PHIP(1),
      1DUM2(1))
  399 FORMAT(' ',2(T3,8D16.9/))
      N=16
      JN=7
      JM=12
C SKIP DOPPLER COUNTS AT LDUM=1 (FIRST MEASUREMENTS)
      IF(LDUM.EQ.1) GO TO 400
C......PROCESS DOPPLER DATA BY STUVAS'S ALGORYTHM........
      CALL POSTMT( MN1,PHI,MPH,2)
      IF (DOP.EQ.0) GO TO 3061
      DO 302 I=1,2
      DO 302 J=1,16
  302 MB(I,J)=MPH(I,J)+NN1(I,J)
      DO 306 I=1,2
      DO 303 J=1,16
      DUB1=0.0D0
      DUB2=0.0D0
      DO 301 K=1,16
            DUB1=DUB1+MB(I,K)*P(K,J)
  301       DUB2= DUB2+MN1(I,K)*H(K,J)
      DUM1(I,J)=DUB1
  303 DUM2(I,J)=DUB2
      DO 306 L=1,2
      SUM=0.0D0
      DO 305 J=1,16
  305 SUM=SUM+DUM1(I,J)*MB(L,J)+DUM2(I,J)*MN1(L,J)
```

```
  306 QN(I,L)=SUM+VN1(I,L)
C COMPUTE  QN INVERSE =QNI
      DUB=QN(1,1)*QN(2,2)-QN(2,1)*QN(1,2)
      QNI(1,1)= QN(2,2)/DUB
      QNI(1,2)= - QN(1,2)/DUB
      QNI(2,1)=-QN(2,1)/DUB
      QNI(2,2)= QN(1,1)/DUB
 3061 CALL PREMT(PHI,P,PHIP,16)
C COMPUTE PHI TRANSPOSE
      DO 365 I=1,16
      DO 365 J=1,16
  365 PHIT(I,J)=PHI(J,I)
      CALL POSTMT(PHIP,PHIT,PPPT,16)
      IF (DOP.EQ.0) GO TO 380
      DO 350 I=1,16
      DO 350 J=1,2
      SUM1=0.0D0
      DO 345 K=1,16
  345       SUM1= SUM1+PHIP(I,K)*MB(J,K)+H(I,K)*MN1(J,K)
  350 DUM3(I,J)=SUM1
C COMPUTE KALMAN GAIN (DOPPLER )
      DO 360 I=1,16
      DO 360 J=1,2
      SUM1=0.0D0
      DO 355 K=1,2
  355      SUM1= SUM1+DUM3(I,K)*QNI(K,J)
  360 BN1(I,J)= SUM1
C COMPUTE ERROR COVARIANCE MATRIX AFTER USING DOPPLER DATA
  380 DO 387 I=1,16
      IF (DOP.EQ.0) GO TO 3855
      DO 385 J=1,2
      DUM=0.0D0
      DO 384 K=1,2
  384       DUM=DUM +BN1(I,K)* QN(K,J)
  385   BQ(I,J)=DUM
 3855 DO 387 L=1,16
      SUM=0.0D0
      IF (DOP.EQ.0) GO TO 3861
      DO 386 J=1,2
  386       SUM=SUM+   BQ(I,J)*BN1(L,J)
 3861  P(I,L)=-SUM+H(I,L)+PPPT(I,L)
      IF (DABS( P(I,L))-1.0D-25)388,388,387
  388  P(I,L)=0.0D0
  387 CONTINUE
  400 WRITE(6,4990) CLOC,CLOC,CLOC,CLOC,DOP,DOP,DOP,DOP,LDUM
 4990 FORMAT('0',T10,'CLOCKS:',4I1,T30,'DOPPLER:',4I1,T50,
     1'ITER:',I2)
      IF (DOP.EQ.0) GO TO 401
      WRITE(6,399)(P(I,I),I=1,16)
      IF (CLOC.EQ.0) GO TO 430
```

```
C....PROCESS TIME MEASUREMENTS BY STANDARD ALGORYTHM......
  401 DO 405 J=1,16
      DUM=0.0D0
      DO 404 K=1,16
  404       DUM=DUM +  P(J,K)*MN2(1,K)
  405 DUM3(J,1)=DUM
      DOM=0.0D0
      DO 410 K=1,16
  410 DOM= DOM+MN2(1,K)*DUM3(K,1)
      DUM=DOM+VN2
      DO 412 J=1,16
  412 DUM3(J,2)=DUM3(J,1)/DUM
C ERROR COVARIANCE MATRIX AFTER USING CLOCKS DATA
      DO 420 I=1,16
      DO 420 J=1,16
  420 P(I,J)= P(I,J)- DUM3(I,2)*DUM3(J,1)
      DO 440 I=1,16
      DO 440 J=1,I
      P (I,J)=(P (I,J)+P (J,I))/2.0D0
      IF(DABS(P(I,J)).LT.1.0D-25)P(I,J)=0.0D0
      P (J,I)=P (I,J)
  440 CONTINUE
      WRITE(6,399)(P(I,I),I=1,16)
C..........END OF KALMAN COMPUTATIONS.................
C COMPUTE COVARIACES IN FEET**2  UP,NORTH,EAST
  430 DC21=DCOS(POL(2,1))
      DC22=DCOS(POL(2,2))
      POL11=POL(1,1)**2
      POL12=POL(1,2)**2
      COV(1)=P(1,1)
      COV(2)=P(2,2)*POL11
      COV(3)=P(3,3)*POL11*DC21**2
      COV(4)=P(4,4)
      COV(5)=P(5,5)*POL12
      COV(6)=P(6,6)*POL12*DC22**2
      COV(7)=P(1,1)+P(4,4)-2*P(4,1)
      COV(8)=(P(2,2)+P(5,5)-2*P(5,2))*POL11
      COV(9)=(P(3,3)+P(6,6)-2*P(6,3))*POL11*DC21**2
C COMPUTE STANDARD DEVIATIONS IN FEET UP,NORTH,EAST
      DO 450 I=1,9
  450 STA(I)=DSQRT(COV(I))
      WRITE(6,499)(STA(I),I=1,9)
  499 FORMAT(' ',3(3(3X,D23.16)/))
      WRITE(7,4991) STA,AC
 4991 FORMAT(1X,15A4)
      RETURN
      END
C
C
C
```

```
C
      SUBROUTINE COEF
C SUBROUTINE COMPUTES GEOMETRY PARAMETERS
      IMPLICIT  REAL*8(A-H,O-Z),INTEGER(I-N)
      DIMENSION C(3,3)
      COMMON P(3,3),A(3,3,3)
      KD=0
      JD=0
      DO 52  J=1,3
      DO 51  K=1,3
C NOW SKIP NON USED J,K PAIRS
      IF(J.EQ.K) GO TO 51
      IF (J+K.EQ.3) GO TO 51
C EARTH XYZ COORDINATES OF K
      IF(K.EQ.KD) GO TO 35
      KD=K
      DC2K=DCOS(P(2,K))
      X=P(1,K)*DC2K          *DCOS(P(3,K))
      Y=P(1,K)*DC2K          *DSIN(P(3,K))
      Z=P(1,K)*DSIN(P(2,K))
C DIRECTION COSINES OF J W/R EARTH XYZ
   35 IF (J.EQ.JD) GO TO 45
      JD=J
      DS2J=DSIN(P(2,J))
      DS3J=DSIN(P(3,J))
      DC2J=DCOS(P(2,J))
      DC3J=DCOS(P(3,J))
      C(2,1)=-DS2J*DC3J
      C(2,2)=-DS2J*DS3J
      C(2,3)=DC2J
      C(3,1)=DS3J
      C(3,2)= -DC3J
      C(3,3)=0.0D0
      C(1,1)=DC2J*DC3J
      C(1,2)=DC2J*DS3J
      C(1,3)=DS2J
   45 DO 50 I=1,3
      RC=(C(I,1)*X+C(I,2)*Y+C(I,3)*Z)/P(1,K)
      IF(I.GT.1) GO TO 40
      RHO= DSQRT(P(1,J)*P(1,J)+P(1,K)*P(1,K)-2.0D0*P(1,J)*
     1P(1,K)*RC)
   40 IF(I-2) 48,47,46
   47 A(I,J,K)=-P(1,J)*P(1,K)*RC/RHO
      GO TO 50
   46 A(I,J,K)= P(1,J)*P(1,K)*RC/RHO
      GO TO 50
   48 A(I,J,K)=(P(1,J)-P(1,K)*RC)/RHO
   50 CONTINUE
   51 CONTINUE
   52 CONTINUE
```

```
      RETURN
      END
C
C
C
C
      SUBROUTINE POSTMT(A,B,P,M)
C SUBROUTINE TO POSTMULTIPLY BY SPARSE MATRIX
      IMPLICIT  REAL*8(A-H,O-Z),INTEGER(I-N)
      COMMON  /MAT/N,JN,JM
C POST MULTIPLY BY ALMOST DIAGONAL MATRIX P=A*B
      DIMENSICN  A(M,N),B(N,N),P(M,N)
      DO 20  I=1,M
      DO 20 J=1,N
    2 P(I,J)=C.CDO
      IF(J.LT.JN) GO TO 5
      IF(J.LE.JM) GO TC 15
    5 P(I,J)=A(I,J)*B(J,J)
      GO TO 20
   15 SUM=0.OC0
      DO 16  K=JN,JM
   16    SUM= SUM+A(I,K)*B(K,J)
      P(I,J)= SUM
   20 CONTINUE
      RETURN
      END
C
C
C
C
      SUBROUTINE  PREMT(A,B,P,M)
C SUBROUTINE TO  PREMULTIPLY BY SPARSE MATRIX
      IMPLICIT  REAL*8(A-H,O-Z),INTEGER(I-N)
      COMMON  /MAT/N,JN,JM
C PREMULTIPLY BY ALMOST DIAGONAL MATRIX
      DIMENSICN A(N,N),B(N,M),P(N,M)
      DO 20  I=1,N
      DO 20  J=1,M
    2 P(I,J)=0.CD0
      IF(I.LT.JN) GO TO 5
      IF(I.LE.JM) GO TO 15
    5 P(I,J)= A(I,I)*B(I,J)
      GO TO 20
   15 SUM=0.OC0
      DO 16  K=JN,JM
   16     SUM=SUM+ A(I,K)*B(K,J)
       P(I,J)=SUM
   20 CONTINUE
      RETURN
      END
```